Reality as a Consensus Protocol: The Fixed-Point Computation That Implements Physics

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Abstract

This paper gives the computational spine of Observer-Patch Holography. A universe is modeled as a finite network of observer patches that may disagree on their overlaps. The network is abstract patch geometry. Spherical, A_5-icosahedral, and E_8-type language in the surrounding OPH stack names regulator and symmetry structure. Local repair moves try to reconcile those disagreements. When the repair law lowers the declared mismatch score, the total inconsistency Φ becomes a Lyapunov functional and termination follows on the finite patch net. Uniqueness is a separate confluence claim: the local-diamond condition on the physical quotient, together with repair completeness, is what makes the terminal physical normal form schedule-independent from a fixed initial quotient state. A stronger same-boundary conclusion requires one more hypothesis: the fixed boundary/sector fiber has a unique consistent quotient extension. The paper also identifies the loop obstructions that block global consistency, explains why physical uniqueness lives on gauge-equivalence classes, carries the same normal-form and holonomy data through separated cofinal refinement systems, shows when coarse-graining and reconciliation commute up to controlled error, gives a fixed-point account of stable records, and proves a conditional fair-block theorem for noisy approximate consensus once a separate contraction certificate is supplied. This is the patch-net theorem package behind the relativity, gauge, Yang–Mills repair-gap, particle, and observer branches.

1 Introduction

This paper writes the OPH consensus picture in the simplest concrete form. A universe is represented as a finite graph of observer patches. Each patch carries local state data, neighboring patches compare those data on overlaps, and local repair moves try to reconcile any mismatch. The central mathematical question is whether this repair dynamics converges to one shared world and how the unavoidable obstructions are encoded when it does not.

The resulting theorem package has two layers. The bare overlap network is only a finite constraint code: its codewords are the globally overlap-consistent states. Quantum error-correction, code-distance / min-cut resilience, exponential mixing, wall-clock BFT liveness, and hardware speedup are separate certified branches with additional data. The first core consensus layer concerns convergence: the declared local-fit contract on touched overlaps makes Φ a Lyapunov functional for accepted repair moves, so termination is derived on the finite patch net; the fixed-cutoff union-collar gluing package then turns competing local repairs into a local diamond on the physical quotient, so under repair completeness every initial quotient state has a unique schedule-independent normal form. The stronger claim that all interiors with the same boundary data settle to the same state requires preservation of that boundary data plus a unique consistent extension in the corresponding fiber. The second concerns obstructions: pairwise overlap agreement does not ensure a global solution, and the obstruction is holonomic. On the abelian branch it is the cycle sum of edge data; on the genuinely noncentral branch it is a crossed-module Čech class.

Gauge symmetry enters as invariance under changes of hidden local representation that preserve overlap data. When the repair step is read only on that overlap-invariant quotient, the normal-form map descends to the gauge quotient, so physical uniqueness is a quotient statement. On the quantum lift, the same quotient-local carrier determines a unique terminal state on every declared physical observable algebra, even when microscopic representative lifts differ by gauge or sector relabelings inside one quotient-local glued state. The observation layer is carried by finite observer-accessible record algebras generated by central or quantitatively stable approximately commuting projectors. These results form the patch-net formulation used in the broader OPH literature, including the fixed-cutoff Bell/CHSH package on the companion microphysics surface.

This paper proves a constraint-code firewall and six core consensus results:

1. **Constraint-code firewall.** A finite overlap net defines a finite constraint code: its codewords are exactly the globally consistent states, and $C = \Phi^{-1}(0)$ (Proposition 2.3). This is the default meaning of "the overlap network is a code." It is not, by itself, a quantum error-correcting code and does not imply a graph min-cut formula for distance (Theorem C.10).
2. **Asynchronous confluence.** For the declared accepted repair law, the local-fit contract makes Φ a Lyapunov functional and hence gives termination, while the fixed-cutoff union-collar gluing package gives the local diamond on the physical quotient; under repair completeness, every fixed initial quotient state has a unique normal form, independent of update schedule (Theorem 3.15).
3. **Cycle obstruction.** For affine overlap constraints over an abelian group, global consistency holds if and only if the holonomy vanishes on every cycle (Theorem 4.1). The parity triangle gives the minimal frustrated example, and Theorem 4.4 extends the same logic to the crossed-module higher-gauge defect hierarchy used later in the framework.
4. **Gauge quotient and observable-level confluence.** When local repair is induced on the overlap-invariant quotient, the normal-form map descends to that quotient, and the induced terminal state on every declared physical observable algebra is unique there even when microscopic representatives differ by gauge or sector relabelings inside one quotient-local glued state. If a boundary/sector map is preserved and each consistent boundary fiber has at most one quotient extension, then all initial states with that boundary value settle to the same quotient normal form (Theorems 5.2, 5.3, 5.6).
5. **Refinement-limit consensus classes.** On a separated cofinal refinement system whose restriction maps commute with the finite-stage normal-form and holonomy maps, the quotient normal forms and holonomy obstructions assemble into unique inverse-limit classes with finite-stage visibility (Theorem 6.2).
6. **Coarse-graining compatibility.** On any refinement system whose coarse-graining maps shadow finite-stage normal forms and holonomy maps with declared errors, reconciling first and then coarse-graining gives the same macroscopic law data as coarse-graining first and then reconciling, up to those errors; in the exact natural case the error is zero (Theorem 6.5).

7. **Record algebra and stability.** On the fixed-cutoff observer-accessible surface, central record projectors carry Born/Lüders measurement directly, and approximate record projectors inherit explicit $(\varepsilon, \delta_{\mathrm{rec}})$ stability bounds on the same event surface (Theorem 7.2).

8. **Conditional noisy fair-block consensus.** If a noisy asynchronous implementation admits fair repair blocks, a uniform expected contraction toward the exact quotient normal-form set, and controlled within-block excursions, then all long runs stay in a controlled expected tube around that exact normal-form set. In singleton boundary/sector fibers, Lipschitz observer readouts are approximately schedule-independent, and a finite Markov-kernel certificate checks the block contraction on finite exported nets (Theorem 8.1, Corollaries 8.2–8.3, and Proposition 8.4).

The repair step itself is a concrete recovery move, not a free rewrite primitive. On the fixed-cutoff collar branch, a local update is obtained from exact Markov splice or from a declared Petz/Fawzi–Renner recovery channel and then read on overlap-invariant physical data. The fixed-point theorem below isolates the separate branch conditions cleanly: the declared repair law includes the touched-overlap local-fit contract that yields Lyapunov descent of Φ on accepted moves, while the fixed-cutoff gluing package supplies the parenthesization-independent union-collar glue used for the local diamond. What sits above that step is repair completeness and, on the Petz branch, the support/CPTP clause stated later in Proposition C.5. A nontrivial exported rooted-tree packet domain where these clauses are proved is recorded in Definition 3.6 and Theorem 3.7.

We also define a fitness functional over a finite candidate space of reconciliation laws and prove that replicator dynamics monotonically increases mean fitness (Theorem 9.4). This gives a clean mathematical model for finite-candidate law selection, not a universality theorem or a literal cosmological dynamics claim.

The results here are exact theorems about a computational model. The companion OPH manuscript uses separate status categories for structural theorems, scaling limits, quantitative particle outputs, and phenomenological continuations [1].

2 Patch Nets, Overlaps, and Global Consistency

Definition 2.1 (Patch net)**.** *Let $G = (V, E)$ be a finite connected graph. Each vertex $i \in V$ is an* ***observer patch*** *with finite local state space S_i. The global state space is*

$$\Sigma := \prod_{i \in V} S_i.$$

For each edge $e = \{i, j\} \in E$, let I_e be an interface alphabet and let

$$\pi_{i,e} : S_i \to I_e, \qquad \pi_{j,e} : S_j \to I_e$$

be the interface projection maps. A global state $s = (s_i)_{i \in V} \in \Sigma$ is ***consistent on edge*** *$e = \{i, j\}$ iff*

$$\pi_{i,e}(s_i) = \pi_{j,e}(s_j).$$

The global consistency set is

$$C := \{s \in \Sigma : \forall\, e = \{i, j\} \in E,\ \pi_{i,e}(s_i) = \pi_{j,e}(s_j)\}.$$

For exposition we use a finite pairwise-overlap graph. The hypergraph version is straightforward: replace edges by hyperedges and pairwise equality by a common interface label on each hyperedge. Nothing in the proofs depends on the pairwise restriction.

The picture: each observer holds a local state, and neighboring observers share an interface through which they can compare notes. A universe-state is physically admissible exactly when all neighbors agree on their shared data. This is a constraint satisfaction problem (CSP), and the consistent states are the codewords.

Definition 2.2 (Inconsistency potential)**.** *For each edge e, choose a weight $w_e > 0$ and a function $d_e : I_e \times I_e \to \mathbb{R}_{\geq 0}$ with $d_e(a, b) = 0 \iff a = b$. On the declared fixed-cutoff branch, d_e is the overlap score used by the local acceptance contract on that interface. Define*

$$\Phi(s) := \sum_{e=\{i,j\}\in E} w_e\, d_e\big(\pi_{i,e}(s_i), \pi_{j,e}(s_j)\big).$$

Then $s \in C \iff \Phi(s) = 0$.

Proposition 2.3 (Bare overlap nets are finite constraint codes)**.** *For every finite patch net of Definition 2.1, the consistency set $C \subseteq \Sigma$ is a finite constraint code whose codewords are exactly the globally overlap-consistent states. With the mismatch potential of Definition 2.2,*

$$C = \Phi^{-1}(0).$$

This proposition is the theorem-grade content of the unqualified phrase "the overlap network is a code." It supplies a finite constraint code, not a quantum error-correcting code, not a topological code, and not a graph-theoretic formula for code distance.

Proof. Finiteness follows from $\Sigma = \prod_i S_i$ with finite S_i. The definition of C is a finite family of interface-equality constraints, so C is the set of satisfying assignments. Since each $w_e > 0$ and $d_e(a, b) = 0$ exactly when $a = b$, every term in Φ is nonnegative and vanishes exactly on a satisfied edge constraint. Therefore all edge constraints hold if and only if $\Phi(s) = 0$. □

So Φ is the total disagreement energy of the universe. Consistent states have zero energy. Everything else is frustrated.

3 Asynchronous Reconciliation and the Main Fixed-Point Theorem

Definition 3.1 (Recovery-derived local repair law)**.** *Fix for each patch i a finite collar chart A_i–B_i–D_i around the overlaps touched by i, together with a fixed local decoder from repaired collar data back to the finite patch label at i. A* ***law*** *λ is a family of local repair maps*

$$T_i^\lambda : \Sigma \to \Sigma \qquad (i \in V)$$

such that T_i^λ changes only the state of patch i (or, more generally, only a bounded neighborhood of i), and the local update is induced by one of the declared OPH recovery moves on that collar, where $\omega_{A_iB_i}(s)$ denotes the $A_i \cup B_i$ marginal of the input collar state encoded by s:

(i) exact Markov splice on the collar, using Theorem A.1 when $I(A_i : D_i \mid B_i) = 0$; or

(ii) a declared recoverability channel

$$(\mathrm{id}_{A_i} \otimes \mathcal{R}_i)(\omega_{A_iB_i}(s)), \qquad \mathcal{R}_i = \mathcal{R}_{\sigma_i,\mathcal{N}_i},$$

with $\mathcal{R}_i$ in the Petz/Fawzi–Renner class of Definition C.3 and Theorem A.1.

The decoder back to S_i is bookkeeping for the finite patch presentation; the physical content is the repaired collar state on the declared fixed-cutoff branch. Write $s \to_i t$ iff $t = T_i^\lambda(s) \neq s$. Let $\to := \bigcup_{i \in V} \to_i$, and let $\to^*$ be its reflexive-transitive closure. A state $s \in \Sigma$ is a ***normal form*** *iff $T_i^\lambda(s) = s$ for all $i \in V$.*

Definition 3.2 (Touched-overlap potential and accepted local-fit contract)**.** *For each repair site i, let*

$$E_i^{\mathrm{touch}} := \{e \in E : \textit{the interface data on } e \textit{ may change under } T_i^\lambda\}.$$

Define the touched-overlap potential

$$\Phi_i(s) := \sum_{e=\{u,v\} \in E_i^{\mathrm{touch}}} w_e\, d_e\big(\pi_{u,e}(s_u), \pi_{v,e}(s_v)\big).$$

On the declared fixed-cutoff branch, a recovery-derived candidate is committed only if it strictly lowers this touched-overlap score:

$$s \to_i t \implies \Phi_i(t) < \Phi_i(s).$$

This is the patch-net form of the regulator-side monotone local-fit contract carried by the declared repair package.

Definition 3.3 (Overlap-associative union-collar gluing)**.** *Fix $s \in \Sigma$ and two enabled repairs $s \to_i t$, $s \to_j u$. Write*

$$E_{ij}^{\mathrm{touch}} := E_i^{\mathrm{touch}} \cup E_j^{\mathrm{touch}}.$$

The declared repair branch is ***overlap-associative*** *if the following hold.*

(i) If $E_i^{\mathrm{touch}} \cap E_j^{\mathrm{touch}} = \varnothing$, the two accepted local updates have disjoint support on the declared branch and therefore commute.

(ii) If $E_i^{\mathrm{touch}} \cap E_j^{\mathrm{touch}} \neq \varnothing$, there is a finite union collar U_{ij} covering the interfaces in E_{ij}^{touch} such that the physical glued state on U_{ij} is parenthesization-independent on the quotient, in the sense of Proposition B.8, and the local decoders/lifts of Definition 3.1 are restriction-compatible on nested collars.

This is the concrete compatibility package used below to complete the local diamond.

Remark 3.4 (Inputs and branch conditions)*.* The repair step is therefore not an abstract rewrite primitive. Its declared inputs are the fixed-cutoff collar chart, either exact Markov splice or a chosen Petz/Fawzi–Renner recovery channel, a local decoder/lift back to the finite patch presentation, and the touched-overlap local-fit contract of Definition 3.2, together with the support-local disjoint-commutation clause and the restriction-compatible union-collar package of Definition 3.3. The parenthesization-independent quotient-local glue used there is supplied by Proposition B.8 from the fixed-cutoff center-sector / higher-gauge gluing package. The theorem package takes repair completeness as an explicit branch condition. On the Petz branch, full CPTP action on all inputs also requires the support clause recorded in Proposition C.5. On broader branches one must also prove that the declared union-collar compatibility is preserved under refinement or branch change.

The theorem package separates the imported repair-law data from the theorem-local inputs cleanly:

Assumption 3.5 (Repair completeness)**.** $s \in C \iff \forall\, i \in V,\ T_i^{\lambda}(s) = s$.

Normal forms are exactly the globally consistent states. The dynamics is neither too weak (missing some inconsistencies) nor too strong (repairing things that were fine).

Definition 3.6 (Verified rooted-tree packet-net domain)**.** *Fix a finite rooted tree $T = (V, E, r)$. For each non-root vertex i, write $p(i)$ for its parent and write $w_i > 0$ for the weight of the edge $\{p(i), i\}$. Let A be a finite packet alphabet with $|A| \geq 2$, and let K_i be a finite hidden-label set. The patch state space is*

$$S_i = A \times K_i, \qquad s_i = (x_i, k_i).$$

For every edge $e = \{i, j\}$, the interface alphabet is $I_e = A$, and both endpoint projections read the packet component:

$$\pi_{i,e}(x_i, k_i) = x_i, \qquad \pi_{j,e}(x_j, k_j) = x_j.$$

Thus e is consistent exactly when $x_i = x_j$. Choose the weights so that

$$w_i > \sum_{j : p(j) = i} w_j \qquad \text{for every non-root } i,$$

with an empty sum equal to 0. Define

$$\Phi(s) = \sum_{\{p(i), i\} \in E} w_i\, \mathbf{1}[x_i \neq x_{p(i)}].$$

The repair map at a non-root vertex i is

$$T_i(s)_i = (x_{p(i)}, k_i),$$

with all other vertices unchanged; if $x_i = x_{p(i)}$, the map is a no-op. The root repair map is the identity. The hidden labels k_i are acted on by arbitrary finite gauge relabelings and are not read by any interface projection.

Theorem 3.7 (Rooted-tree packet repair completeness and quotient closure)**.** *On the domain of Definition 3.6, Assumption 3.5 is a theorem. Every enabled repair strictly decreases Φ. Every maximal asynchronous repair run terminates at the unique state*

$$x_i = x_r \quad \text{for all } i \in V,$$

with all hidden labels k_i unchanged. The normal-form map descends to the quotient by hidden gauge relabeling. The four-vertex instance with edges $r - a$, $a - b$, $a - c$, alphabet $A = \mathbb{Z}_3$, hidden labels $K_i = \mathbb{Z}_2$, and weights $5, 1, 1$ is exported as `code/consensus/runs/verified_tree_packet_net_domain.json`.

Proof. First, repair completeness is immediate from the rooted tree. If $s \in C$, every edge has $x_i = x_{p(i)}$, so every non-root repair is a no-op. Conversely, if every repair is a no-op, then $x_i = x_{p(i)}$ for every non-root vertex, hence every edge is consistent and $s \in C$.

Let $i \neq r$ be enabled. Only the parent edge $\{p(i), i\}$ and child edges $\{i, j\}$ with $p(j) = i$ can change their contribution to Φ. The parent edge changes from inconsistent to consistent, contributing $-w_i$. Each child edge can increase by at most its weight. Therefore

$$\Phi(T_i(s)) - \Phi(s) \leq -w_i + \sum_{j : p(j) = i} w_j < 0.$$

So every enabled repair is accepted by the Lyapunov contract.

Since Φ takes finitely many values, every maximal repair run terminates. At a terminal state no non-root vertex differs from its parent, so the terminal packet label is x_r on every vertex. The root label and every hidden label are invariant under all repairs, so the terminal state is unique and independent of update order. Because interface projections, enabledness, Φ, and the repair maps depend only on x_i, arbitrary relabelings of the hidden K_i commute with quotienting:

$$q \circ T_i = \overline{T}_i \circ q.$$

Thus the normal-form map descends to the hidden-label quotient. $\square$

Corollary 3.8 (Physical-law map on the verified packet domain)**.** *On the rooted-tree packet domain, every gauge-invariant observable*

$$M : \prod_i (A \times K_i) \to Y$$

has a schedule-independent repaired value

$$M(\mathrm{nf}(s)),$$

and this value depends only on the quotient class of s. Hence the promoted normal-form map is usable as physical law on this verified packet branch without adding a representative-level gauge-covariance assumption.

Proof. Theorem 3.7 gives a unique terminal state for every asynchronous repair schedule and shows that the normal-form map descends to the quotient by hidden-label relabeling. A gauge-invariant observable factors through that quotient, so its value on the terminal state is independent of both the repair schedule and the hidden representative. $\square$

Proposition 3.9 (Classical full-support Petz packet domain)**.** *Let B and D be finite packet alphabets, let the collar algebras be diagonal, and let $\mathcal{N} : \mathbb{C}^{B\times D} \to \mathbb{C}^{B}$ be the marginal channel $(\mathcal{N}p)(b) = \sum_d p(b,d)$. Fix a reference state σ_{BD} with $\sigma_B(b) > 0$ for every b. Let*

$$\gamma_\sigma := \min_{b\in B} \sigma_B(b) > 0.$$

Define the Petz recovery channel

$$\mathcal{R}_{\sigma,\mathcal{N}}(\mu)(b,d) = \mu(b)\,\sigma(d \mid b), \qquad \sigma(d \mid b) := \frac{\sigma_{BD}(b,d)}{\sigma_B(b)}.$$

Then $\mathcal{R}_{\sigma,\mathcal{N}}$ is stochastic, completely positive and trace preserving on the diagonal algebra, and ℓ^1-contractive:

$$\|\mathcal{R}_{\sigma,\mathcal{N}}(\mu) - \mathcal{R}_{\sigma,\mathcal{N}}(\nu)\|_1 \le \|\mu - \nu\|_1.$$

The support inverse is uniformly bounded on this domain by

$$\|\sigma_B^{-1/2}\| \le \gamma_\sigma^{-1/2}.$$

If a collar state has the exact classical Markov form

$$\omega_{ABD}(a,b,d) = \omega_{AB}(a,b)\sigma(d \mid b),$$

then $(\mathrm{id}_A \otimes \mathcal{R}_{\sigma,\mathcal{N}})(\omega_{AB}) = \omega_{ABD}$. The support obstruction is exact: if $\sigma_B(b) = 0$ and an input assigns mass to b, the Petz inverse on that sector is undefined unless the channel domain is restricted or a separate trace-preserving completion is declared.

Proof. The displayed formula is the finite diagonal specialization of the Petz map

$$\sigma_{BD}^{1/2}\left(\sigma_B^{-1/2}\mu\,\sigma_B^{-1/2}\otimes\mathbf{1}_D\right)\sigma_{BD}^{1/2}.$$

Full support of σ_B makes the inverse well defined, with support gap $\gamma_\sigma > 0$, hence $\|\sigma_B^{-1/2}\| \leq \gamma_\sigma^{-1/2}$. Nonnegativity is immediate, and

$$\sum_{b,d}\mathcal{R}_{\sigma,\mathcal{N}}(\mu)(b,d) = \sum_b \mu(b)\sum_d \sigma(d\mid b) = \sum_b \mu(b),$$

so the map is trace preserving. Diagonal positive trace-preserving maps are completely positive on the diagonal algebra. For signed diagonal inputs,

$$\|\mathcal{R}_{\sigma,\mathcal{N}}(\mu) - \mathcal{R}_{\sigma,\mathcal{N}}(\nu)\|_1 = \sum_{b,d}|\mu(b)-\nu(b)|\,\sigma(d\mid b) = \sum_b |\mu(b)-\nu(b)|,$$

which gives the stated contraction. The exact Markov recovery identity follows by substitution. If $\sigma_B(b) = 0$, the factor $\sigma_B^{-1/2}$ has no value on that sector; mass placed there by an input is outside the Petz support domain. That is the claimed obstruction. ☐

Proposition 3.10 (Accepted repair moves are Lyapunov-decreasing)**.** *For the accepted repair law of Definitions 3.1 and 3.2, every enabled repair strictly decreases the inconsistency potential:*

$$s \to t \implies \Phi(t) < \Phi(s).$$

Proof. Write $s \to_i t$. If $e \notin E_i^{\mathrm{touch}}$, then the interface data on e are unchanged by T_i^λ, so the corresponding term in Φ is the same for s and t. Therefore

$$\Phi(t) - \Phi(s) = \Phi_i(t) - \Phi_i(s) < 0$$

by Definition 3.2. ☐

Proposition 3.11 (Termination from the OPH Lyapunov functional)**.** *Under Proposition 3.10, every repair sequence is finite; equivalently, the repair relation $\to$ is terminating.*

Proof. Because each S_i is finite, the global state space $\Sigma = \prod_{i\in V} S_i$ is finite, so the value set $\Phi(\Sigma) \subset \mathbb{R}_{\geq 0}$ is finite as well. Along any nontrivial repair step $s \to t$, Proposition 3.10 gives $\Phi(t) < \Phi(s)$. An infinite repair sequence would therefore produce an infinite strictly descending chain in the finite ordered set $\Phi(\Sigma)$, which is impossible. ☐

Proposition 3.12 (Finite repair step bound)**.** *Let $s_0 \in \Sigma$. Under Proposition 3.10, every accepted repair run starting at s_0 has length at most*

$$\left|\{\,\Phi(s) : s\in\Sigma,\ \Phi(s)\leq\Phi(s_0)\,\}\right| - 1 \leq |\Phi(\Sigma)| - 1.$$

If, additionally, Φ is integer-valued, or more generally every accepted move lowers Φ by at least a fixed $\eta > 0$, then the run length satisfies

$$T(s_0) \leq \left\lceil \frac{\Phi(s_0)}{\eta} \right\rceil.$$

This is a finite descent bound in repair steps only. It is not a wall-clock bound, not a probability-one scheduling statement, and not spectral or exponential convergence.

Proof. The values of Φ strictly decrease along the run, so no value in $\{\, \Phi(s) : s \in \Sigma,\ \Phi(s) \le \Phi(s_0) \,\}$ can occur twice. This gives the finite value-set bound. If each move lowers Φ by at least η, after T moves the value has dropped by at least $T\eta$. Since $\Phi \ge 0$, $T\eta \le \Phi(s_0)$, giving the displayed ceiling bound. ☐

Remark 3.13 (Termination is not confluence). Proposition 3.11 proves only that accepted repair runs stop. It does not prove that two update orders stop at the same physical state. Schedule-independence enters only after the local-diamond property of Proposition 3.14 is combined with termination via Newman's lemma and with repair completeness in Assumption 3.5. If two accepted repair schedules from the same initial state reach different observer-facing quotient normal forms, with no declared holonomy or higher-gauge obstruction and no mere gauge-representative difference, then the proposed repair law is not OPH-admissible as a consensus mechanism.

Proposition 3.14 (Local confluence from overlap-associative gluing)**.** *For the accepted repair law of Definitions 3.1, 3.2, and 3.3, the repair relation is locally confluent.*

Proof. Take $s \to_i t$ and $s \to_j u$. If $E_i^{\text{touch}} \cap E_j^{\text{touch}} = \varnothing$, then Definition 3.3(i) gives

$$T_j^\lambda(T_i^\lambda(s)) = T_i^\lambda(T_j^\lambda(s)).$$

So with $v := T_j^\lambda(T_i^\lambda(s))$ we have $t \to_j v$ and $u \to_i v$.

Assume $E_i^{\text{touch}} \cap E_j^{\text{touch}} \neq \varnothing$. Let U_{ij} be the union collar from Definition 3.3(ii), and let $\overline{v}_{ij}(s)$ denote the quotient-local glued state on U_{ij} determined by the exterior marginals of s. Proposition B.8 makes this state independent of whether the local splice/recovery is parenthesized as i then j or j then i. Because the local decoders are restriction-compatible on nested collars, the one-site repairs $s \to_i t$ and $s \to_j u$ are precisely the i- and j-restrictions of $\overline{v}_{ij}(s)$. Applying that same restriction-compatible decoder to the unfinished subcollar therefore produces the complementary accepted local step from t and from u into representatives of the same quotient-local union-collar state. Hence there exists $v \in \Sigma$ with $t \to^* v$ and $u \to^* v$, where v is any representative of $\overline{v}_{ij}(s)$. ☐

Theorem 3.15 (Asynchronous confluence / fixed-point law)**.** *For the accepted repair law of Definitions 3.1, 3.2, and 3.3, under Assumption 3.5, each fixed initial state $s \in \Sigma$ has a unique normal form*

$$\mathrm{nf}_\lambda(s) \in C,$$

and every maximal asynchronous repair execution from s terminates at that same state. The terminal state is independent of update order because descent, local confluence on the physical quotient, and repair completeness all hold; termination alone is not enough.

Proof. By Proposition 3.11, the repair relation $\to$ is terminating. By Proposition 3.14, it is locally confluent. Newman's lemma [2] therefore makes it confluent. A terminating confluent repair relation has a unique normal form reachable from each fixed initial state. By Assumption 3.5, the normal forms are exactly C. Every maximal execution terminates and reaches $\mathrm{nf}_\lambda(s)$, and that terminal state is independent of update order. ☐

Corollary 3.16 (Objective law is schedule-independent)**.** *Let $M : \Sigma \to Y$ be any observable. Under the hypotheses of Theorem 3.15, $M(\mathrm{nf}_\lambda(s))$ is independent of the asynchronous update schedule. If physical law is identified with the map $s \mapsto M(\mathrm{nf}_\lambda(s))$, then physical law is objective.*

Proof. All schedules from the same initial s terminate at $\mathrm{nf}_\lambda(s)$, so all yield the same M-value. ☐

Here objectivity is identified with schedule-independent convergence of the repair dynamics; Theorem 5.6 sharpens this to schedule-independent convergence of the physical observable algebra even when microscopic representatives are not unique.

Remark 3.17 (Quantifier on normal-form uniqueness)*.* Theorem 3.15 proves uniqueness from a fixed initial state, and Theorem 5.2 below turns that into uniqueness from a fixed physical quotient state. It does not say that all possible initial data settle to one universal state. Different boundary conditions, conserved charges, root packets, holonomy sectors, or external records can legitimately determine different normal forms.

4 Why Local Agreement Is Not Enough: Cycle Holonomy and Frustration

Pairwise neighbor agreement does not by itself imply global consistency.

Theorem 4.1 (Cycle-obstruction / holonomy criterion)**.** *Let A be an abelian group, and let $G = (V, E)$ be a connected graph with an arbitrary orientation on each edge. For each oriented edge $e : u \to v$, assign a label $b_e \in A$. Consider the affine consistency equations*

$$x_v - x_u = b_e \qquad \text{for every oriented edge } e : u \to v,$$

where $x_v \in A$ are unknown patch labels. A global solution $x : V \to A$ exists if and only if for every cycle $C \subseteq G$,

$$\sum_{e \in C} \varepsilon_C(e)\, b_e = 0,$$

where $\varepsilon_C(e) = +1$ if the cycle traverses e in the chosen orientation and -1 otherwise.

Proof. **Necessity.** Suppose x is a solution. Summing the edge equations around any cycle C,

$$\sum_{e:u \to v \in C} \varepsilon_C(e)\,(x_v - x_u) = \sum_{e \in C} \varepsilon_C(e)\, b_e.$$

The left side telescopes to 0 because every vertex appears once with + sign and once with − sign.

Sufficiency. Fix a root $r \in V$. For any vertex v, choose a path $P_{r \to v}$ and define

$$x_v := \sum_{e \in P_{r \to v}} \varepsilon_{P_{r \to v}}(e)\, b_e, \qquad x_r := 0.$$

If P and P' are two paths from r to v, traversing P followed by the reverse of P' yields a cycle. By the vanishing-holonomy assumption, the total signed sum is zero, so x_v is well-defined. For any edge $e : u \to v$, extending a path to u by that edge gives $x_v = x_u + b_e$. □

Corollary 4.2 (Parity triangle: pairwise consistency is not enough)**.** *Take $A = \mathbb{Z}_2$ on the triangle A-B-C-A with edge labels $b_{AB} = 0$, $b_{BC} = 0$, $b_{CA} = 1$. Each individual edge equation is satisfiable. But the global system is not: the cycle sum is $0 \oplus 0 \oplus 1 = 1 \neq 0$.*

This example shows that pairwise consistency does not imply a global solution. The obstruction is carried by the cycle.

Corollary 4.3 (Stable defects as frustrated holonomy)**.** *Define the defect energy*

$$\Phi_b(x) := \sum_{e:u\to v\in E} w_e\, \mathbf{1}[x_v - x_u \neq b_e]\,.$$

If the cycle-holonomy condition fails, then $\min_{x:V\to A} \Phi_b(x) > 0$*. Every minimizer contains irreducible residual inconsistency.*

Proof. If $\min_x \Phi_b(x) = 0$, some assignment satisfies all edge equations, contradicting Theorem 4.1. ☐

Residual inconsistencies of this type cannot be removed by local repair moves. In the OPH interpretation they are stable topological defects of the reconciliation dynamics.

Theorem 4.4 (Higher-gauge defect hierarchy)**.** *Let a finite overlap nerve carry crossed-module defect data* (g_{ij}, h_{ijk}) *for a compact crossed module*

$$H \xrightarrow{\partial} G.$$

Under local rechartings by

$$C^1(N,H) \rtimes C^0(N,G),$$

the nonabelian Čech class

$$q = [(g,h)] \in \check{H}^2(N, H \to G)$$

is invariant. Strict global reconciliation exists if and only if $q = 0$*, and nonzero* q *labels stable fixed-cutoff higher-gauge defects.*

Proof. The allowed rechartings are exactly the crossed-module coboundaries. They preserve the class and strictify the weak gluing data precisely when the class vanishes. A nonzero class therefore gives a topologically protected residual obstruction in the full crossed-module defect hierarchy, including the abelian truncation as its first layer. ☐

5 Gauge Symmetry as Implementation Hiding

Definition 5.1 (Gauge action)**.** *Let* $\Gamma = \prod_{i\in V} \Gamma_i$ *act on* Σ *componentwise. The action is a* ***gauge action*** *if for every* $e = \{i,j\} \in E$*,*

$$\pi_{i,e}(\gamma_i \cdot x) = \pi_{i,e}(x) \qquad \forall\, x \in S_i,\ \forall\, \gamma_i \in \Gamma_i.$$

Gauge changes alter hidden local representations but do not alter overlap data.

Gauge transformations change hidden local representations while leaving overlap data fixed. Write

$$q : \Sigma \to \Sigma/\Gamma, \qquad q(s) = [s],$$

for the gauge-orbit map. A **physical repair law** is the family of quotient-local maps

$$\overline{T}_i^{\lambda} : \Sigma/\Gamma \to \Sigma/\Gamma$$

induced by the recovery-derived collar updates of Definition 3.1. A **representative repair family** is any choice of lifts

$$T_i^{\lambda} : \Sigma \to \Sigma$$

such that

$$q \circ T_i^\lambda = \overline{T}_i^\lambda \circ q.$$

This is the finite patch-net form of saying that the repair step is defined first on overlap-invariant physical data and only then lifted to hidden representatives. The rooted-tree packet domain of Theorem 3.7 proves repair completeness and quotient descent on a nontrivial exported packet net, while Proposition 3.9 proves the classical full-support Petz clause used by that domain. A broader fixed-cutoff branch requires repair completeness for the chosen exported packet net and the support/CPTP clause on the Petz branch where that channel is used. The touched-overlap local-fit contract gives Lyapunov Φ-descent on accepted moves, and Proposition B.8 together with Definition 3.3 supplies the quotient-local compatibility package used for the local diamond. Stability of that package under refinement or branch change is a separate question. The point here is also that no extra gauge-covariance axiom is needed once repair is formulated on the quotient.

Theorem 5.2 (Gauge quotient theorem)**.** *Under the gauge action of Definition 5.1, any representative lift of a physical repair law as just defined, and the hypotheses of Theorem 3.15,*

$$q\big(\mathrm{nf}_\lambda(\gamma \cdot s)\big) = q\big(\mathrm{nf}_\lambda(s)\big) \qquad \forall\, \gamma \in \Gamma,\ \forall\, s \in \Sigma.$$

Hence the normal-form map descends to the quotient:

$$\overline{\mathrm{nf}}_\lambda : \Sigma/\Gamma \to \Sigma/\Gamma, \qquad [s] \mapsto [\mathrm{nf}_\lambda(s)].$$

Proof. Suppose $s \to_i t$, so $t = T_i^\lambda(s) \neq s$. If $s' = \gamma \cdot s$, then

$$q(T_i^\lambda(s')) = \overline{T}_i^\lambda\big(q(s')\big) = \overline{T}_i^\lambda\big(q(s)\big) = q\big(T_i^\lambda(s)\big).$$

Thus gauge-equivalent inputs induce the same repaired orbit, and by induction the orbit reached after any repair sequence depends only on the initial orbit. Every maximal repair sequence from s ends at $\mathrm{nf}_\lambda(s)$ by Theorem 3.15, so the terminal orbit $q(\mathrm{nf}_\lambda(s))$ depends only on $q(s) = [s]$. This makes

$$[s] \longmapsto [\mathrm{nf}_\lambda(s)]$$

well-defined on Σ/Γ. □

Theorem 5.3 (Boundary-conditioned quotient uniqueness)**.** *Let*

$$B : \Sigma/\Gamma \to \mathcal{B}$$

record fixed external boundary data, conserved charge, root packet, holonomy sector, or task input. Assume accepted quotient repairs preserve B:

$$[s] \to [t] \implies B([s]) = B([t]).$$

For $b \in \mathcal{B}$, define the consistent quotient fiber

$$C_b := \{\, x \in q(C) : B(x) = b \,\}.$$

If every C_b has at most one element, then all initial states with the same boundary value settle to the same observer-facing quotient normal form:

$$B([s]) = B([s']) \implies \overline{\mathrm{nf}}_\lambda([s]) = \overline{\mathrm{nf}}_\lambda([s']).$$

Proof. Theorem 5.2 gives unique quotient normal forms $\overline{\mathrm{nf}}_\lambda([s])$ and $\overline{\mathrm{nf}}_\lambda([s'])$ in $q(C)$. Boundary preservation along accepted repair sequences gives

$$B(\overline{\mathrm{nf}}_\lambda([s])) = B([s]), \qquad B(\overline{\mathrm{nf}}_\lambda([s'])) = B([s']).$$

If $B([s]) = B([s']) = b$, both quotient normal forms lie in C_b. By the unique consistent extension assumption, C_b has at most one element, so the quotient normal forms are equal. □

Corollary 5.4 (Gauge-invariant law)**.** *If $M : \Sigma \to Y$ is gauge-invariant ($M(\gamma \cdot s) = M(s)$ for all γ), then $M(\mathrm{nf}_\lambda(s))$ depends only on the gauge orbit $[s]$, not the representative.*

Proof. Because M is gauge-invariant, it factors through the orbit map: $M = \overline{M} \circ q$ for some $\overline{M} : \Sigma/\Gamma \to Y$. Theorem 5.2 gives

$$q(\mathrm{nf}_\lambda(\gamma \cdot s)) = q(\mathrm{nf}_\lambda(s)),$$

hence

$$M(\mathrm{nf}_\lambda(\gamma \cdot s)) = \overline{M}(q(\mathrm{nf}_\lambda(\gamma \cdot s))) = \overline{M}(q(\mathrm{nf}_\lambda(s))) = M(\mathrm{nf}_\lambda(s)).$$

□

Definition 5.5 (Physical observable algebra on the quantum lift)**.** *Fix a finite patch region or union collar R on the declared fixed-cutoff quantum lift of Appendix B. Its* ***physical observable algebra*** *$\mathcal{A}_{\mathrm{phys}}(R)$ is the fixed-point collar algebra under the compact boundary redundancy action on the ordinary or central-defect branch, or the corresponding quotient-local algebra on the genuinely noncentral branch. The central sector projectors carried by the collar decomposition belong to $Z(\mathcal{A}_{\mathrm{phys}}(R))$. A* ***physical observable*** *is any $X \in \mathcal{A}_{\mathrm{phys}}(R)$. For a microscopic representative $s \in \Sigma$, let ω_R^s denote the induced state on $\mathcal{A}_{\mathrm{phys}}(R)$.*

Theorem 5.6 (Observable-level confluence on the quantum lift)**.** *Under the hypotheses of Theorems 3.15 and 5.2, every initial orbit $[s] \in \Sigma/\Gamma$ has a unique quotient normal form*

$$\overline{\mathrm{nf}}_\lambda([s]) \in q(C).$$

Fix a declared region R. Let $t, u \in \Sigma$ be terminal microscopic representatives reached by representative lifts of maximal repair sequences from initial states in the same orbit $[s]$. If the induced R-collar states of t and u are representatives of the same quotient-local glued state in the sense of Proposition B.8, then for every physical observable $X \in \mathcal{A}_{\mathrm{phys}}(R)$,

$$\omega_R^t(X) = \omega_R^u(X).$$

Hence all physical observables converge to the same overlap-consistent values even when the microscopic terminal representatives differ by gauge relabelings globally or by sector/higher-gauge relabelings inside one declared quotient-local glued state.

Proof. Theorems 3.15 and 5.2 give the unique quotient normal form $\overline{\mathrm{nf}}_\lambda([s]) = [\mathrm{nf}_\lambda(s)] \in q(C)$. Let t and u be as stated. By Corollary B.10, representatives of the same quotient-local glued state induce the same state on the physical observable algebra $\mathcal{A}_{\mathrm{phys}}(R)$, including the central sector projectors carried by that algebra. Therefore $\omega_R^t(X) = \omega_R^u(X)$ for every $X \in \mathcal{A}_{\mathrm{phys}}(R)$. □

Remark 5.7 (Inputs and boundary for observable-level confluence)*.* Theorem 5.6 does not add a new repair hypothesis beyond the confluence and fixed-cutoff gluing package. Its inputs are exactly the representative-level confluence theorem, quotient descent of the repair law, Definition 5.5, and Corollary B.10 from the fixed-cutoff union-collar gluing package. The boundary item is extension of the same statement to broader refinement-stable branches where the declared union-collar compatibility is only approximate or where the chosen physical observable algebra itself changes under refinement.

Corollary 5.8 (Inert ancillary refinement does not change physical law)**.** *Let* $K = \prod_i K_i$ *be a finite ancillary state space and define* $\Sigma^\eta := \Sigma \times K$*. Lift the repair maps by*

$$T_i^{\lambda,\eta}(s,k) := (T_i^\lambda(s), k).$$

If $M^\eta : \Sigma^\eta \to Y$ *ignores the ancillary factor,*

$$M^\eta(s,k) = M(s),$$

with M *gauge-invariant on* Σ*, then*

$$M^\eta(\mathrm{nf}^\eta_\lambda(s,k)) = M(\mathrm{nf}_\lambda(s))$$

for all $(s,k) \in \Sigma^\eta$*.*

Proof. Because the ancillary factor is inert,

$$\mathrm{nf}^\eta_\lambda(s,k) = (\mathrm{nf}_\lambda(s), k).$$

Therefore $M^\eta(\mathrm{nf}^\eta_\lambda(s,k)) = M(\mathrm{nf}_\lambda(s))$, and Corollary 5.4 supplies gauge-orbit independence. □

Physical uniqueness therefore holds on the quotient by gauge or implementation hiding. The same statement is unchanged under inert ancillary stabilization.

Definition 5.9 (Finite packet closure simplex)**.** *Assume the finite fixed-cutoff consensus branch of Theorems 3.15 and 5.2, so the physical quotient* $Q := \Sigma/\Gamma$ *is finite and carries the schedule-independent quotient normal-form map*

$$\overline{\mathrm{nf}}_\lambda : Q \to Q.$$

Let

$$N_\lambda := \overline{\mathrm{nf}}_\lambda(Q) = q(C)$$

be the quotient normal-form set. The ***finite packet simplex*** *on* Q *is*

$$\Delta(Q) := \left\{ \mu : Q \to \mathbb{R}_{\geq 0} \;\middle|\; \sum_{x \in Q} \mu(x) = 1 \right\},$$

and the ***finite packet closure map*** *is the pushforward*

$$\mathcal{C}_\lambda : \Delta(Q) \to \Delta(Q), \qquad \mathcal{C}_\lambda(\mu) := \left(\overline{\mathrm{nf}}_\lambda\right)_* \mu,$$

equivalently

$$\mathcal{C}_\lambda(\mu)(y) = \sum_{\substack{x \in Q \\ \overline{\mathrm{nf}}_\lambda(x) = y}} \mu(x).$$

Theorem 5.10 (Finite packet-quotient closure map)**.** *On the finite fixed-cutoff consensus branch of Definition 5.9, the map $\mathcal{C}_\lambda$ is an affine continuous idempotent self-map of $\Delta(Q)$. Its image is exactly the normal-form simplex*

$$\Delta(N_\lambda) = \{\mu \in \Delta(Q) \mid \operatorname{supp}(\mu) \subseteq N_\lambda\}\,.$$

Hence the fixed points of $\mathcal{C}_\lambda$ are exactly the packets supported on quotient normal forms. For every initial quotient state $[s] \in Q$,

$$\mathcal{C}_\lambda(\delta_{[s]}) = \delta_{\overline{\mathrm{nf}}_\lambda([s])}.$$

Proof. Because $\overline{\mathrm{nf}}_\lambda : Q \to Q$ is a set map on a finite set, its pushforward on probability packets is affine and continuous in the finite-dimensional simplex topology.

By Theorems 3.15 and 5.2, the quotient normal form is schedule-independent and terminal, so

$$\overline{\mathrm{nf}}_\lambda \circ \overline{\mathrm{nf}}_\lambda = \overline{\mathrm{nf}}_\lambda.$$

Pushforward therefore gives

$$\mathcal{C}_\lambda^2 = \left(\overline{\mathrm{nf}}_\lambda\right)_*\left(\overline{\mathrm{nf}}_\lambda\right)_* = \left(\overline{\mathrm{nf}}_\lambda \circ \overline{\mathrm{nf}}_\lambda\right)_* = \mathcal{C}_\lambda,$$

so $\mathcal{C}_\lambda$ is idempotent.

If $\nu = \mathcal{C}_\lambda(\mu)$, then every point in $\operatorname{supp}(\nu)$ lies in the image of $\overline{\mathrm{nf}}_\lambda$, namely N_λ. Thus $\operatorname{im}(\mathcal{C}_\lambda) \subseteq \Delta(N_\lambda)$. Conversely, if $\nu \in \Delta(N_\lambda)$, then $\overline{\mathrm{nf}}_\lambda(y) = y$ for every $y \in N_\lambda$, so

$$\mathcal{C}_\lambda(\nu) = \nu.$$

Hence $\Delta(N_\lambda) \subseteq \operatorname{im}(\mathcal{C}_\lambda)$, proving $\operatorname{im}(\mathcal{C}_\lambda) = \Delta(N_\lambda)$. The same identity shows that ν is a fixed point if and only if it is supported on N_λ. The Dirac-mass formula is the pushforward of a point mass under $\overline{\mathrm{nf}}_\lambda$. ☐

Remark 5.11 (Boundary of the finite closure theorem)*.* Theorem 5.10 is an exact finite-branch result. It proves a closure map only on the finite packet simplex built from one fixed quotient carrier whose repair law is terminating, confluent, and quotient-descended. It does not prove a habitat-level closure map for arbitrary OPH state-and-law data, does not identify a nonempty observer-supporting invariant sector inside the Appendix-B habitat, and does not supply uniqueness or stability estimates beyond this finite packet branch.

6 Refinement-Limit Consensus Classes

The finite consensus theorem is the operational object used by the continuum branches. To pass from one finite patch net to a refining family, the paper uses an explicit inverse-limit package with named compatibility clauses.

Definition 6.1 (Separated cofinal refinement consensus system)**.** *Let $(R, \preceq)$ be a directed refinement set. For each $r \in R$, let*

$$Q_r := \Sigma_r/\Gamma_r$$

be the finite physical quotient state space of a patch presentation, let

$$n_r : Q_r \to Q_r$$

be the finite-stage quotient normal-form map supplied by Theorems 3.15 and 5.2, and let

$$h_r : Q_r \to \mathcal{H}_r$$

be the finite-stage holonomy or higher-gauge obstruction map supplied by Theorems 4.1 and 4.4. For $r \preceq s$, assume restriction maps

$$\rho_{sr} : Q_s \to Q_r, \qquad \chi_{sr} : \mathcal{H}_s \to \mathcal{H}_r,$$

with $\rho_{rr} = \mathrm{id}$, $\chi_{rr} = \mathrm{id}$, and the cocycle identities

$$\rho_{tr} = \rho_{sr} \circ \rho_{ts}, \qquad \chi_{tr} = \chi_{sr} \circ \chi_{ts} \qquad (r \preceq s \preceq t).$$

The system is a separated cofinal refinement consensus system when:

1. ***Normal-form naturality:***

$$\rho_{sr} \circ n_s = n_r \circ \rho_{sr} \qquad (r \preceq s).$$

2. ***Holonomy naturality:***

$$\chi_{sr} \circ h_s = h_r \circ \rho_{sr} \qquad (r \preceq s).$$

3. ***Visible separation:*** *two compatible families in $\varprojlim Q_r$, or in $\varprojlim \mathcal{H}_r$, are equal whenever their projections agree on a cofinal subset of stages.*

Theorem 6.2 (Refinement-limit consensus and holonomy classes)**.** *For any separated cofinal refinement consensus system, the formulas*

$$n_\infty((x_r)_r) := (n_r(x_r))_r, \qquad h_\infty((x_r)_r) := (h_r(x_r))_r$$

define maps

$$n_\infty : \varprojlim Q_r \to \varprojlim Q_r, \qquad h_\infty : \varprojlim Q_r \to \varprojlim \mathcal{H}_r.$$

The class $n_\infty(x)$ is the unique schedule-independent refinement-limit normal form of x. The class $h_\infty(x)$ is the refinement-limit holonomy obstruction. The pair $(n_\infty(x), h_\infty(x))$ is the refinement-limit consensus class. In the inverse-limit topology, finite-stage normal forms and holonomy classes converge to these two classes: for every stage r, all refinements $s \succeq r$ restrict to the fixed values

$$\rho_{sr}(n_s(x_s)) = n_r(x_r), \qquad \chi_{sr}(h_s(x_s)) = h_r(x_r).$$

Furthermore, $h_\infty(x) = 0$ if and only if every finite projection has zero finite-stage obstruction. If $h_\infty(x) \neq 0$, visible separation gives a finite stage that witnesses the nonzero obstruction. If two refinement-limit candidates have the same normal-form and holonomy projections on a cofinal tail, then they determine the same refinement-limit consensus class.

Proof. Let $x = (x_r)_r \in \varprojlim Q_r$, so $\rho_{sr}(x_s) = x_r$ for $r \preceq s$. Normal-form naturality gives

$$\rho_{sr}(n_s(x_s)) = n_r(\rho_{sr}(x_s)) = n_r(x_r),$$

so $(n_r(x_r))_r$ is a compatible family and n_∞ is well defined. The same argument with holonomy naturality gives

$$\chi_{sr}(h_s(x_s)) = h_r(\rho_{sr}(x_s)) = h_r(x_r),$$

so h_∞ is well defined.

Each finite $n_r(x_r)$ is independent of update schedule by Theorems 3.15 and 5.2. Therefore every finite projection of $n_\infty(x)$ is schedule-independent, and visible separation makes the inverse-limit class unique. The displayed restriction identities are exactly the cylinder convergence statement in the inverse-limit topology.

The zero-obstruction statement follows from the definition of the inverse-limit zero class. The identity $h_\infty(x) = 0$ holds exactly when every projection $h_r(x_r)$ is zero. If $h_\infty(x) \neq 0$, at least one finite projection is nonzero, and any cofinal tail containing a refinement of that stage carries the same nonzero projected obstruction by holonomy naturality. The last claim is visible separation applied to the compatible normal-form and holonomy families. □

Remark 6.3 (Scope of the refinement theorem). Theorem 6.2 is a controlled continuum bridge. It proves persistence, exhaustion, and collapse of the consensus data once the OPH refinement system supplies the restriction maps and the two naturality clauses in Definition 6.1. It leaves uniform complexity bounds, automatic fair-block noisy-consensus certificates, and automatic normal-form naturality for arbitrary changing repair laws as separate branch conditions.

Definition 6.4 (Controlled coarse-graining / reconciliation square)**.** *Fix refinement stages $r \preceq s$. Let Q_r, Q_s be the physical quotient state spaces, n_r, n_s their finite-stage normal-form maps, and h_r, h_s their holonomy or higher-gauge obstruction maps. Let*

$$\rho_{sr} : Q_s \to Q_r, \qquad \chi_{sr} : \mathcal{H}_s \to \mathcal{H}_r$$

be the coarse-graining maps on quotient states and obstruction data. Equip Q_r and $\mathcal{H}_r$ with pseudometrics d_r^Q and $d_r^{\mathcal{H}}$ that define the macroscopic readout scale at stage r.

The square is $(\varepsilon_{sr}^n, \varepsilon_{sr}^h)$-controlled when, for every $x \in Q_s$,

$$d_r^Q(\rho_{sr}(n_s(x)),\, n_r(\rho_{sr}(x))) \le \varepsilon_{sr}^n,$$

and

$$d_r^{\mathcal{H}}(\chi_{sr}(h_s(x)),\, h_r(\rho_{sr}(x))) \le \varepsilon_{sr}^h.$$

The errors are ***cofinally vanishing*** *when, for every fixed macroscopic stage r and every $\delta > 0$, there is a refinement stage $s_0 \succeq r$ such that $\varepsilon_{sr}^n, \varepsilon_{sr}^h < \delta$ for all $s \succeq s_0$.*

Theorem 6.5 (Coarse-graining commutes with reconciliation up to controlled error)**.** *Suppose the refinement square of Definition 6.4 is $(\varepsilon_{sr}^n, \varepsilon_{sr}^h)$-controlled. Define the coarse-stage consensus readout*

$$\mathcal{C}_r(y) := (n_r(y), h_r(y))$$

and the fine-then-coarse readout

$$\mathcal{C}_{s\to r}^{\mathrm{fine}}(x) := (\rho_{sr}(n_s(x)), \chi_{sr}(h_s(x))).$$

Then, for every fine state $x \in Q_s$, the product readout distance between $\mathcal{C}_{s\to r}^{\mathrm{fine}}(x)$ and $\mathcal{C}_r(\rho_{sr}(x))$ is bounded by

$$\max\{\varepsilon_{sr}^n, \varepsilon_{sr}^h\}.$$

Thus reconciling at the fine stage and then coarse-graining gives the same macroscopic law data as coarse-graining first and reconciling at the coarse stage, up to the declared control errors. If the exact naturality clauses of Definition 6.1 hold, then $\varepsilon_{sr}^n = \varepsilon_{sr}^h = 0$. If the errors are cofinally vanishing, the two procedures determine the same cylinder values in the inverse-limit topology at every fixed macroscopic stage.

Proof. The normal-form component of the product readout is bounded by the first inequality in Definition 6.4, and the obstruction component is bounded by the second. Taking the maximum gives the stated product bound. Exact naturality is precisely the special case

$$\rho_{sr} \circ n_s = n_r \circ \rho_{sr}, \qquad \chi_{sr} \circ h_s = h_r \circ \rho_{sr},$$

so both errors vanish there. Cofinal vanishing means that, for any fixed coarse cylinder and any desired readout tolerance, all sufficiently fine presentations give the same cylinder value within that tolerance, which is convergence in the inverse-limit topology. □

Remark 6.6 (What remains to verify on a concrete RG branch). Theorem 6.5 is not a claim that arbitrary renormalization maps commute with arbitrary repair laws. It identifies the exact branch burden: derive or estimate the two square defects ε^n_{sr} and ε^h_{sr} from the chosen coarse-graining channel, recovery map, decoder, and obstruction readout. The exact refinement theorem is the zero-defect case; approximate RG matching is theorem-grade only when these defects are explicitly controlled.

7 Record Algebras and the Operator Observation Layer

Inputs used here. From the fixed-cutoff collar package we use only the observer-accessible finite-dimensional algebra on one completed compare/write/verify slice, the declared pointer and overlap-sector projectors read on that same slice, and the trace-distance control on restored accessible states when restoration is invoked. No continuum lift and no broader observer-metaphysical assumption is used here.

Definition 7.1 (Exact record algebras and approximate record presentations)**.** *Fix one completed observer-accessible slice at cycle t, and let $\mathcal{A}^{\mathrm{acc}}_t$ be the corresponding finite-dimensional accessible algebra. An* exact record presentation *is a family of orthogonal projectors $\{\widehat{P}_a(t)\}_a \subset Z(\mathcal{A}^{\mathrm{acc}}_t)$ whose generated algebra*

$$\mathcal{Z}_{\mathrm{rec}}(t) := \mathrm{Alg}(\{\widehat{P}_a(t)\}_a)$$

is finite and commutative.

An approximate record presentation *on the same declared readout slots is a family of projectors $\{P_a(t)\}_a \subset \mathcal{A}^{\mathrm{acc}}_t$ together with an exact record presentation $\{\widehat{P}_a(t)\}_a$ such that*

$$\delta_{\mathrm{rec}}(t) := \max_a \|P_a(t) - \widehat{P}_a(t)\|$$

is finite.

Theorem 7.2 (Record algebra, Born-Lüders update, and quantitative stability)**.** *Fix one completed observer-accessible slice at cycle t.*

1. *The exact record projectors generate a finite commutative central algebra $\mathcal{Z}_{\mathrm{rec}}(t) \subset Z(\mathcal{A}^{\mathrm{acc}}_t)$.*
2. *For every event E in the finite event algebra generated by $\mathcal{Z}_{\mathrm{rec}}(t)$,*

$$\mathbb{P}_t(E) = \mathrm{Tr}\big(\rho_t \widehat{P}_E(t)\big), \qquad \rho_t|_E = \frac{\widehat{P}_E(t)\rho_t\widehat{P}_E(t)}{\mathrm{Tr}(\rho_t \widehat{P}_E(t))}.$$

3. *If no accepted repair between t and $t+1$ touches the support of $\widehat{P}_E(t)$, then the next read of E has probability* 1.

4. If $\{P_a(t)\}_a$ is an approximate record presentation with modulus $\delta_{\mathrm{rec}}(t)$, then

$$\|[P_a(t), P_b(t)]\| \le 4\,\delta_{\mathrm{rec}}(t)$$

for all declared record projectors $P_a(t), P_b(t)$. Moreover, for every declared elementary record event a and every restored accessible state $\widetilde{\rho}_t$ satisfying

$$\|\widetilde{\rho}_t - \rho_t\|_1 \le \varepsilon,$$

one has

$$\Big|\mathrm{Tr}\big(\widetilde{\rho}_t P_a(t)\big) - \mathrm{Tr}\big(\rho_t \widehat{P}_a(t)\big)\Big| \le \varepsilon + \delta_{\mathrm{rec}}(t).$$

Proof. Because the exact record projectors lie in the center of $\mathcal{A}_t^{\mathrm{acc}}$, they generate a finite commutative central algebra. Finite-dimensional projective measurement on that commuting algebra gives the Born trace and the Lüders conditioned state, proving (1) and (2).

If no accepted repair touches the support of $\widehat{P}_E(t)$, then the next completed read is performed on the same central projector surface. After conditioning on E, the state lies in the range of $\widehat{P}_E(t)$, so the next read of the same event has probability 1. This gives (3).

For (4), write

$$[P_a(t), P_b(t)] = [P_a(t) - \widehat{P}_a(t), P_b(t)] + [\widehat{P}_a(t), P_b(t) - \widehat{P}_b(t)],$$

because $[\widehat{P}_a(t), \widehat{P}_b(t)] = 0$. Since every projector has operator norm at most 1,

$$\|[P_a(t), P_b(t)]\| \le 2\|P_a(t) - \widehat{P}_a(t)\| + 2\|P_b(t) - \widehat{P}_b(t)\| \le 4\,\delta_{\mathrm{rec}}(t).$$

For the probability bound,

$$\Big|\mathrm{Tr}\big(\widetilde{\rho}_t P_a(t)\big) - \mathrm{Tr}\big(\rho_t \widehat{P}_a(t)\big)\Big| \le \Big|\mathrm{Tr}\big((\widetilde{\rho}_t - \rho_t) P_a(t)\big)\Big| + \Big|\mathrm{Tr}\big(\rho_t (P_a(t) - \widehat{P}_a(t))\big)\Big|.$$

The first term is bounded by $\|\widetilde{\rho}_t - \rho_t\|_1 \|P_a(t)\| \le \varepsilon$, and the second by $\|\rho_t\|_1 \|P_a(t) - \widehat{P}_a(t)\| \le \delta_{\mathrm{rec}}(t)$. Hence the total error is at most $\varepsilon + \delta_{\mathrm{rec}}(t)$. ☐

Merge boundary. What is closed here is the fixed-cutoff operator-algebraic observation surface: a central record algebra on the exact readout slice, Born/Lüders measurement on its event projectors, exact repeated-read stability under the untouched-support hypothesis, and explicit $(\varepsilon, \delta_{\mathrm{rec}})$ control when practical readout projectors are only close to that central surface. The broader export problem asks whether richer branches keep physically relevant pointer surfaces close to one such central record algebra and whether the same control survives refinement and continuation.

8 Noisy Fair-Block Approximate Consensus

The exact consensus theorem supplies the quotient normal-form target. A noisy implementation needs one more quantitative certificate: complete fair blocks must contract expected distance to that target. This section records the conditional bridge from local noisy repair to long-run observer-facing approximate consensus.

Theorem 8.1 (Global noisy approximate consensus under fair-block contraction)**.** *Let (Q, d_Q) be the observer-facing quotient state space of a fixed exported OPH patch federation, and let $\mathcal{N} \subset Q$ be the exact quotient normal-form set supplied by the finite repair theorem on that quotient. Define*

$$D(q) := d_Q(q, \mathcal{N}) = \inf_{n \in \mathcal{N}} d_Q(q, n).$$

Let $q_{t+1} = \widetilde{T}_{i_t,t}(q_t)$ be a noisy asynchronous repair process adapted to a filtration $(\mathcal{F}_t)_t$. Assume:

- *(G1)* ***Exact quotient target.*** *The ideal repair relation on Q has the exact OPH normal-form package: finite descent, quotient-local confluence, and repair completeness, so every fixed initial quotient state has a schedule-independent exact normal form in $\mathcal{N}$.*
- *(G2)* ***Fair asynchronous blocks.*** *There are stopping times*

  $$0 = \tau_0 < \tau_1 < \tau_2 < \cdots$$

  such that each interval $[\tau_m, \tau_{m+1})$ contains enough local repairs to expose and act on every active mismatch class required by the exact local-diamond and repair-completeness certificate, with uniformly bounded length $\tau_{m+1} - \tau_m \leq L$. Write the noisy block map as

  $$\widetilde{B}_m := \widetilde{T}_{i_{\tau_{m+1}-1},\, \tau_{m+1}-1} \circ \cdots \circ \widetilde{T}_{i_{\tau_m},\, \tau_m}.$$

- *(G3)* ***Uniform block contraction toward normal form.*** *There are constants $0 < \lambda < 1$ and $\varepsilon \geq 0$ such that, for every block m and every reachable quotient state q at time τ_m,*

  $$\mathbb{E}\Big[D\Big(\widetilde{B}_m(q)\Big) \mid \mathcal{F}_{\tau_m}\Big] \leq \lambda D(q) + \varepsilon.$$

- *(G4)* ***Controlled within-block excursions.*** *There are constants $A \geq 1$ and $\beta \geq 0$ such that, for every $t \in [\tau_m, \tau_{m+1})$,*

  $$\mathbb{E}[D(q_t) \mid \mathcal{F}_{\tau_m}] \leq A\, D(q_{\tau_m}) + \beta.$$

Then, at fair-block times,

$$\mathbb{E}[D(q_{\tau_m})] \leq \lambda^m D(q_0) + \frac{1 - \lambda^m}{1 - \lambda}\, \varepsilon,$$

and hence

$$\limsup_{m \to \infty} \mathbb{E}[D(q_{\tau_m})] \leq \frac{\varepsilon}{1 - \lambda}.$$

At all intermediate asynchronous times,

$$\limsup_{t \to \infty} \mathbb{E}[D(q_t)] \leq A\, \frac{\varepsilon}{1 - \lambda} + \beta.$$

Thus the noisy asynchronous OPH repair process converges in expectation to a controlled tube around the exact quotient normal-form set. If $\varepsilon = \beta = 0$, then $D(q_t) \to 0$ in L^1, hence also in probability, along the full asynchronous run.

Proof. Let $D_m := D(q_{\tau_m})$. Assumption (G3), applied to the realized state q_{τ_m}, gives

$$\mathbb{E}[D_{m+1} \mid \mathcal{F}_{\tau_m}] \leq \lambda D_m + \varepsilon.$$

Taking expectations,

$$\mathbb{E}[D_{m+1}] \leq \lambda\, \mathbb{E}[D_m] + \varepsilon.$$

Iterating the scalar recursion gives

$$\mathbb{E}[D_m] \le \lambda^m D(q_0) + \varepsilon \sum_{r=0}^{m-1} \lambda^r = \lambda^m D(q_0) + \frac{1-\lambda^m}{1-\lambda}\varepsilon.$$

Taking $m \to \infty$ yields the fair-block limsup bound. For $t \in [\tau_m, \tau_{m+1})$, Assumption (G4) gives

$$\mathbb{E}[D(q_t)] \le A\,\mathbb{E}[D(q_{\tau_m})] + \beta.$$

Substitution of the fair-block estimate and then taking the limsup over all intermediate times gives

$$\limsup_{t\to\infty} \mathbb{E}[D(q_t)] \le A\,\frac{\varepsilon}{1-\lambda} + \beta.$$

If $\varepsilon = \beta = 0$, then $\mathbb{E}[D(q_{\tau_m})] \le \lambda^m D(q_0) \to 0$. Since $D \ge 0$, convergence in expectation to zero implies convergence in probability at block times. Assumption (G4) then gives $\mathbb{E}[D(q_t)] \le A\,\mathbb{E}[D(q_{\tau_m})]$ inside each block, so the same L^1 and probability convergence holds along the full asynchronous run. ☐

Corollary 8.2 (Approximate observer-facing schedule independence)**.** *Let $M : Q \to Y$ be an observer-facing readout into a metric space (Y, d_Y), and suppose M is L_M-Lipschitz. Fix an exact sector ζ whose exact normal-form set is the singleton $\mathcal{N}_\zeta = \{n_\zeta\}$, and apply Theorem 8.1 on that sector, so that $D(q) = d_Q(q, n_\zeta)$. Then*

$$\limsup_{t\to\infty} \mathbb{E}[d_Y(M(q_t), M(n_\zeta))] \le L_M \left(A\,\frac{\varepsilon}{1-\lambda} + \beta \right).$$

For two noisy asynchronous schedules q_t and q'_t started in the same exact singleton sector and satisfying the same certificate,

$$\limsup_{t\to\infty} \mathbb{E}\big[d_Y(M(q_t), M(q'_t))\big] \le 2L_M \left(A\,\frac{\varepsilon}{1-\lambda} + \beta \right).$$

Proof. The Lipschitz condition gives

$$d_Y(M(q_t), M(n_\zeta)) \le L_M d_Q(q_t, n_\zeta) = L_M D(q_t).$$

The first bound follows from Theorem 8.1. The two-schedule bound follows from the triangle inequality through $M(n_\zeta)$ and applying the first estimate to both schedules. ☐

Corollary 8.3 (High-probability noisy consensus tube)**.** *Assume the block-distance process also admits the pathwise decomposition*

$$D_{m+1} \le \lambda D_m + \varepsilon + \xi_{m+1}, \qquad \mathbb{E}[\xi_{m+1} \mid \mathcal{F}_{\tau_m}] = 0, \qquad |\xi_{m+1}| \le b$$

almost surely. Then, for every $a > 0$,

$$\Pr\Big[D_m > \lambda^m D_0 + \frac{1-\lambda^m}{1-\lambda}\varepsilon + a\Big] \le \exp\!\left(-\frac{a^2(1-\lambda^2)}{2b^2}\right).$$

Proof. Unrolling the recursion gives

$$D_m \le \lambda^m D_0 + \frac{1-\lambda^m}{1-\lambda}\varepsilon + \sum_{r=1}^{m} \lambda^{m-r}\xi_r.$$

The final term is a weighted martingale sum with increments bounded by $|\lambda^{m-r}\xi_r| \le \lambda^{m-r}b$. Azuma–Hoeffding gives

$$\Pr\left[\sum_{r=1}^{m} \lambda^{m-r}\xi_r > a\right] \le \exp\left(-\frac{a^2}{2\sum_{r=1}^{m} \lambda^{2(m-r)} b^2}\right).$$

Since $\sum_{r=1}^{m} \lambda^{2(m-r)} \le (1-\lambda^2)^{-1}$, substitution yields the claimed bound. ☐

Proposition 8.4 (Finite fair-block contraction certificate)**.** *Let* Q *be finite, let* $\mathfrak{B}_{\mathrm{fair}}$ *be a finite list of noisy fair-block types, and let* $K_B(q,q')$ *be the Markov kernel induced by block type* B*. If there are constants* $0 < \lambda < 1$ *and* $\varepsilon \ge 0$ *such that*

$$\sum_{q' \in Q} K_B(q,q')D(q') \le \lambda D(q) + \varepsilon \qquad \forall q \in Q, \quad \forall B \in \mathfrak{B}_{\mathrm{fair}},$$

then Assumption (G3) of Theorem 8.1 holds for any run whose fair blocks are drawn from $\mathfrak{B}_{\mathrm{fair}}$*.*

Proof. Conditioning on the current quotient state q and the realized fair-block type B, the conditional expectation of D after the block is exactly $\sum_{q'} K_B(q,q')D(q')$. The displayed inequality is therefore Assumption (G3), uniformly over all reachable states and fair-block types. ☐

Finite audit route and constants. For a finite exported packet net, Proposition 8.4 gives the practical certificate path:

finite quotient $Q \Rightarrow$ exact normal forms $\mathcal{N} \Rightarrow$ distance table $D(q) \Rightarrow$ noisy fair-block kernels $K_B \Rightarrow (\lambda, \varepsilon)$ certifica

Here $\mathcal{N}$ is the exact quotient normal-form set, $D(q)$ is observer-facing residual distance to that set, λ is the net contraction produced by one completed fair block, ε is the per-block irreducible local recovery / record / readout / calibration / environmental noise, A bounds transient within-block expansion, β is the within-block noise floor, and L is the fairness horizon before all required active mismatch classes are serviced. In quantum/collar implementations, ε may absorb Petz-domain truncation, Fawzi–Renner recovery error, approximate central-record error, detector/readout noise, and coarse-graining defect.

Claim boundary. Theorem 8.1 is a conditional global noisy-consensus theorem. It does not follow from the exact finite repair theorem alone. The exact OPH theorem supplies the quotient normal-form target $\mathcal{N}$; the noisy theorem requires a separate fair-block contraction certificate $(\lambda, \varepsilon, A, \beta, L)$ for the chosen implementation. Without that certificate, OPH retains exact fixed-cutoff convergence and the collar-local splice and record-stability estimates above. With that certificate, arbitrarily long asynchronous noisy repair sequences converge to a controlled observer-facing tube around the exact quotient normal-form set.

9 Law-Space Selection and Observer Emergence

This section studies a simple meta-selection model on law space. The aim is to formalize one criterion for favoring schedule-robust, observer-supporting, and simple laws; the replicator dynamics below is part of the model, not a claim about literal cosmological dynamics.

We begin by defining what it means for a law to support observers. An observer is treated operationally as a **persistent predictive module**: a subgraph that maintains a stable record algebra and uses its output law to predict its boundary's future behavior.

Definition 9.1 (Schedule robustness)**.** *Fix distributions μ over initial conditions and ν over asynchronous schedules, and a gauge-invariant observable M. For a law λ, define*

$$\mathcal{R}_M(\lambda) := \Pr_{s\sim\mu,\,\sigma,\tau\sim\nu}\left[M(\mathrm{nf}^{\sigma}_{\lambda}(s)) = M(\mathrm{nf}^{\tau}_{\lambda}(s))\right].$$

If Theorem 3.15 holds for λ, then $\mathcal{R}_M(\lambda) = 1$.

Definition 9.2 (Observer yield)**.** *Let X_t^{λ} denote the stationary process obtained by repeated local perturbation plus reconciliation under law λ. For each subgraph $U \subseteq V$, let $\mathcal{Z}_U^{\mathrm{rec}}(t)$ be the declared exact record algebra on its observer-accessible surface, or the reference exact algebra when only an approximate record presentation is available, and let $Y_U(t)$ be the corresponding finite outcome variable induced by that record algebra. Then U is (η, ε, h)-observer-like if it is record-stable:*

$$d_{\mathrm{TV}}(\mathrm{Law}(Y_U(t+h)), \mathrm{Law}(Y_U(t))) \le \eta,$$

and predictive:

$$I(Y_U(t);\ X^{\lambda}_{\partial U,\,t+1:t+h}) \ge \varepsilon.$$

Define

$$\mathcal{O}_{\eta,\varepsilon,h}(\lambda) := \mathbb{E}[\#\{U \subseteq V : U \text{ is } (\eta,\varepsilon,h)\text{-observer-like}\}].$$

Definition 9.3 (Law fitness)**.** *Let $K(\lambda)$ be a description-length penalty. Define*

$$f(\lambda) = \alpha\,\mathcal{R}_M(\lambda) + \beta\,\mathcal{O}_{\eta,\varepsilon,h}(\lambda) - \gamma\,K(\lambda),$$

with $\alpha, \beta, \gamma > 0$.

Theorem 9.4 (Replicator monotonicity on law space)**.** *Let $\Lambda = \{\lambda_1, \ldots, \lambda_m\}$ be candidate laws with population weights $x_i(t)$ under replicator dynamics:*

$$\dot{x}_i = x_i(f_i - \bar{f}), \qquad f_i := f(\lambda_i), \qquad \bar{f} := \sum_{j=1}^{m} x_j f_j.$$

Then

$$\frac{d}{dt}\bar{f} = \mathrm{Var}_x(f) \ge 0.$$

Mean fitness is nondecreasing, and strictly increasing unless all extant laws have the same fitness.

Proof. Direct computation:

$$\frac{d}{dt}\bar{f} = \sum_i \dot{x}_i f_i = \sum_i x_i(f_i - \bar{f}) f_i = \sum_i x_i f_i^2 - \bar{f}^2 = \mathrm{Var}_x(f) \ge 0.$$

Equality iff all f_i on the support of x are equal. □

This theorem records the monotonicity property of the meta-selection model.

10 Connection to Observer-Patch Holography

The formalism above is the computational skeleton of Observer-Patch Holography (OPH). The observer patches carry von Neumann algebras on support-visible holographic cuts. In symmetric regulator charts those cuts may be represented by patches on a screen S^2. The fixed-cutoff microphysics carrier is a federated patch system with echosahedral local interfaces. The S^2 chart supplies cap and collar geometry, and in the companion relativity branch its conformal group supplies the Lorentz bridge. The carrier supplies finite ports, records, and repair interfaces. The overlap projections are restrictions to shared subalgebras, and the consistency condition is algebraic state agreement on overlaps.

The bridge to physics works as follows in the broader companion corpus:

- The patch net becomes a net of support-visible subregion algebras; S^2 is the standard observer-facing regulator chart, not a required literal substrate.
- Overlap Consistency, one of the canonical framework axioms, is the algebraic version of Definition 2.1.
- The recoverability clause of the canonical Recoverable Generalized Entropy axiom provides controlled collar recovery data. The exact collar factorization $\rho_{ABD} = \bigoplus_\alpha p_\alpha \, \rho^{(\alpha)}_{Ab^\alpha_L} \otimes \rho^{(\alpha)}_{b^\alpha_R D}$ is used only at exact Markovity or along the fixed-collar replacement limit, while the declared Petz/Fawzi–Renner recovery channels supply recovered comparison states from which the local repair moves are built (see Appendix A and Definition C.3).
- Gauge symmetry as implementation hiding (Theorem 5.2) becomes the fixed-cutoff edge-sector package from which the compact-gauge branch, using the theorem-produced coherent refinement transport and compatible finite fibers, reconstructs a compact gauge group from zero-obstruction sectors. Zero obstruction is the transportability/classification condition; it is not the Standard Model selector. The realized witness theorem supplies nonempty MAR-admissible branch data, and MAR plus the explicit one-Higgs matter package then selects the Standard Model quotient $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)/\mathbb{Z}_6$ together with the exact hypercharge lattice and the realized counts $N_c = 3$, $N_g = 3$.
- On the declared support-visible compact-gauge branch, the companion compact paper obtains the four-dimensional Euclidean Yang–Mills form from compact-gauge holonomy data and the local MaxEnt/Gibbs continuum limit, with the branch assumptions stated there. It then applies the same quotient-first repair logic to Euclidean transfer: active exact-Markov repairs become conditional expectations, their finite-stage generator has a uniform repair gap, and coherent continuum extraction identifies the Yang–Mills gap with the repair gap. The Clay-facing admissibility claim is exactly that this branch-local support-visible extraction and its Osterwalder–Schrader reconstruction supply the required four-dimensional axiomatic construction.
- The coarse-graining compatibility theorem (Theorem 6.5) is the link between the finite reconciliation protocol and the refinement/RG language used by the OPH branches: the macroscopic law space is stable when the selected coarse-graining channel shadows the normal-form and obstruction maps with controlled defects.
- Stable defects (Corollary 4.3) become the topologically protected excitations identified with particles on the declared branch.

- The record-algebra theorem (Theorem 7.2) provides the formal basis for the fixed-cutoff observation layer, where records are carried by central or quantitatively stable approximately commuting projectors in overlap centers.

The companion manuscripts develop a derived gravity branch from entanglement equilibrium and modular geometry, the SM gauge-group and count closure from edge-sector reconstruction plus the realized MAR branch, a support-visible compact-gauge Yang–Mills form and repair-gap theorem under the declared branch assumptions, and a controlled large-N_{edge} worldsheet effective-description branch above the heat-kernel edge-sector identity. The implemented cosmological capacity branch, with its zero-input self-closure target, and later phenomenological continuations are kept explicitly separate from that recovered-core claim set.

This paper provides the finite patch-net foundation for the broader companion corpus.

11 Discussion and Scope Boundaries

This paper proves the fixed-point consensus spine of OPH: schedule-independent normal forms from fixed initial quotient states, boundary-conditioned uniqueness when a preserved boundary/sector fiber has a unique consistent extension, holonomy obstructions, gauge-quotient invariance, separated cofinal refinement-limit consensus classes, controlled coarse-graining compatibility, the fixed-cutoff operator-record theorem, and conditional noisy fair-block convergence once a separate contraction certificate is supplied. It also proves the full repair-completeness, Petz-domain, and quotient-compatibility package on the rooted-tree packet-net domain of Theorem 3.7, and gives a clean law-selection meta-model. The relativity chain and the realized Standard Model structural chain are the recovered core. The capacity relation is a separate implemented branch with a zero-input self-closure target. Downstream phenomenology requires additional assumptions beyond the consensus results proved here.

Complexity boundary. On a fixed finite patch net, the accepted reconciliation dynamics is a finite-state asynchronous rewrite system on Σ. Under Theorem 3.15, every accepted repair run has at most $|\Phi(\Sigma)| - 1 \leq |\Sigma| - 1$ nontrivial steps, because each accepted move strictly lowers Φ and the value set $\Phi(\Sigma)$ is finite. Exact normal-form computation is therefore decidable by direct iteration of accepted local repairs. This step bound is a termination statement; the schedule-independent answer still depends on the local-diamond and repair-completeness clauses of Theorem 3.15. What this paper does *not* prove is a uniform polynomial-time bound, a sharper complexity-class placement for families of growing patch nets, or any hardness lower bound.

Approximate-stability boundary. The theorem-grade consensus statement is exact on the declared fixed-cutoff branch. Approximate control begins collar-locally: Theorem A.1 gives the exact-Markov modulus $\delta^{\mathrm{M}}_{A:B:D}(\varepsilon) \to 0$ on one fixed finite-dimensional collar model and the one-shot recovery comparison bound $2\sqrt{1 - e^{-\varepsilon}} \leq 2\sqrt{\varepsilon}$, while Theorem 7.2 gives the $(\varepsilon, \delta_{\mathrm{rec}})$ repeated-read stability bound for approximate record projectors. The long-run noisy statement is conditional: Theorem 8.1 upgrades those local noisy controls to a global expected tube around the exact quotient normal-form set only after a fair-block contraction certificate $(\lambda, \varepsilon, A, \beta, L)$ is supplied for the chosen implementation. It does not follow from finite descent or fairness alone, and it gives a unique approximate readout only inside singleton boundary/sector fibers.

Expressive-power boundary. The law-selection model of Theorem 9.4 is a finite-candidate monotonicity result. For each fixed patch net the theorem package proves a finite-state exact reconciliation mechanism. A universality claim would require an explicit uniform family of patch nets and repair laws that simulates arbitrary circuits or machines with stated encoding overhead and robustness under asynchronous schedules. No such theorem is supplied here.

With those boundaries explicit, the consensus-paper boundaries and companion interfaces are:

1. **Sharper RG-shadowing estimates.** Theorem 6.5 closes the abstract reconciliation/coarse-graining square once its normal-form and obstruction defects are supplied. The quantitative task is to derive model-specific or uniform bounds for ε^n_{sr} and ε^h_{sr} from concrete OPH coarse-graining channels and recovery decoders.

2. **Defect classification and refinement-limit transportability.** The fixed-cutoff hierarchy extends from abelian frustrations to crossed-module classes $q \in \check{H}^2(N, H \to G)$. The continuation task is to connect those higher-gauge defect sectors to the refinement-stable transportable sector category used in the broader compact-gauge reconstruction lane.

3. **Observable-level confluence beyond the declared fixed-cutoff physical algebra.** Theorem 5.6 closes the fixed-cutoff quantum-lift statement when microscopic representatives differ by gauge relabelings globally or by sector/higher-gauge relabelings on the same declared quotient-local glued state. The continuation question is whether an analogous observable theorem survives on broader refinement-stable branches where the union-collar compatibility is only approximate or where the physical observable algebra itself changes under refinement.

4. **Global approximate-consensus stability.** Theorem A.1 and Theorem 7.2 supply the collar-local perturbative controls carried by this paper. The rooted-tree packet domain proves the exact finite repair package on one nontrivial exported domain. Theorem 8.1 closes the long-run noisy branch only after a fair-block contraction certificate is supplied. What remains absent is an automatic theorem showing that arbitrary approximate recovery moves on arbitrary packet-closed exported overlap nets are repair-complete, preserve support-local disjoint commutation and nested-collar restriction compatibility, satisfy the support/CPTP clause on every Petz branch used there, and meet that fair-block contraction certificate.

5. **Expressive power / universality.** A universality claim would require an explicit uniform family of patch nets and repair laws that simulates arbitrary circuits or machines with stated encoding overhead and robustness under asynchronous schedules. This paper supplies no such construction, so universality is outside the theorem-grade output.

6. **Interface to the companion gravity/gauge stack.** The companion compact paper derives the gravity chain through the support-visible BW scaling theorem on the prime geometric cap subnet. This consensus paper supplies the finite-state and refinement-consensus spine used by that bridge, while dark-sector proposals, heuristic baryogenesis continuations, spectroscopy, and string/worldsheet topics remain separate continuations that require their own declared inputs beyond what this paper itself proves.

The last item is an interface statement rather than an extra consensus premise. The fixed-point theorems proved here stand on their stated finite and refinement-consensus hypotheses, while companion gravity, gauge, and continuation sectors add their own theorem surfaces and declared inputs.

11.1 Assumption-Dependent BFT and QECC Extensions

The consensus formalism of OPH has natural analogies to classical and quantum distributed Byzantine agreement. Observer patches correspond to protocol nodes, overlap repair corresponds to a quorum vote, and the repair fixed-point corresponds to a consensus state. Under explicit structural assumptions (quorum size $\geq 2f+1$, partial synchrony, one-vote-per-view, certificate semantics, and DLS-style view-change), a QBFT-style interpretation of OPH repair satisfies safety and liveness (Appendix C, Theorem C.2). On the fixed-cutoff collar branch used here, the repair map is written in exact-splice / Petz form; the assumption-dependent item is the CPTP property on all inputs, which requires either full-rank $\mathcal{N}(\sigma)$ or an explicit domain restriction, and trace-preserving completion is not automatic when $\mathcal{N}(\sigma)$ has a non-trivial kernel (Proposition C.5). A quantum error-correcting interpretation is possible only after a genuine code subspace, logical dictionary, error family, and recovery map are supplied. The graph-min-cut equality for distance is not a property of a bare overlap graph: the same graph can realize distance 1 or distance $|V|$ under different interface maps (Theorem C.10). Distance/min-cut statements require the topological-code certificate of Definition C.11 and Theorem C.12; resilience requires the Knill–Laflamme certificate of Theorem C.14. All of these extensions are assumption-dependent or conjectural and are not part of the core theorem package of Paper 4.

Desired statement	**Required certificate**
Overlap network is a code	finite constraint-code data of Proposition 2.3
Repair converges	finite Lyapunov descent; confluence additionally needs local diamond and repair completeness
Distance equals min-cut	topological-code certificate with homological logicals and matching systole/min-cut geometry
Corrects t corrupted carriers	code projector, error family, Knill–Laflamme condition, and certified $t < d/2$ distance bound
Exponential convergence	declared transfer/channel operator with stationary projection and spectral gap
Long-run noisy approximate consensus	fair-block contraction certificate $(\lambda, \varepsilon, A, \beta, L)$ for distance to the exact quotient normal-form set
Wall-clock BFT liveness	partial synchrony, quorum certificates, authentication, and $n \geq 3f+1$
Hardware search work reduction	exact-verifier candidate-enrichment factor measured under controls

A Quantum/Algebraic Lift: Markov-Collar Splice Theorem

This appendix records the algebraic splice statement used to relate the finite patch-net model to the OPH collar formalism.

An observer is written as

$$O = (P, \mathcal{A}(P), \rho, R),$$

where P is the screen patch, $\mathcal{A}(P)$ the local von Neumann algebra, ρ the local state, and R the record algebra.

Theorem A.1 (Markov-collar splice theorem, exact and controlled). *Suppose a collar tripartition A-B-D has exact Markov decomposition*

$$\rho_{ABD} = \bigoplus_{\alpha} p_\alpha \, \rho^{(\alpha)}_{Ab^\alpha_L} \otimes \rho^{(\alpha)}_{b^\alpha_R D}.$$

Let $\sigma^{(\alpha)}_{b^\alpha_R D'}$ *be any family of normalized environment states compatible with the same right-boundary sectors. Define*

$$\rho'_{ABD'} = \bigoplus_\alpha p_\alpha\, \rho^{(\alpha)}_{Ab^\alpha_L} \otimes \sigma^{(\alpha)}_{b^\alpha_R D'}.$$

Then for every observable X *supported on* $A \cup b_L$*,*

$$\operatorname{Tr}(X\rho'_{ABD'}) = \operatorname{Tr}(X\rho_{ABD}).$$

Fix one finite-dimensional collar model and let

$$\mathfrak{M}_{A:B:D} := \{\tau_{ABD} : I(A:D \mid B)_\tau = 0\},$$

with exact-Markov distance modulus

$$\delta^{\mathrm{M}}_{A:B:D}(\varepsilon) := \sup\left\{ \inf_{\tau\in\mathfrak{M}_{A:B:D}} \|\omega-\tau\|_1 : I(A:D\mid B)_\omega \le \varepsilon \right\}.$$

Then

$$\delta^{\mathrm{M}}_{A:B:D}(\varepsilon) \to 0 \qquad (\varepsilon \downarrow 0).$$

On a fixed faithful collar class with lower floor $\lambda_* > 0$*, this qualitative modulus can be sharpened to a collar-local rate*

$$\delta^{\mathrm{M},\lambda_*}_{A:B:D}(\varepsilon) \le C_{A:B:D,\lambda_*}\, \varepsilon^{\theta_{A:B:D,\lambda_*}},$$

by the compact real-analytic Lojasiewicz inequality applied to $I(A:D\mid B)$*. The constants depend on the fixed collar model and floor; they are not a dimension-free stability theorem for arbitrary tripartite systems. Hence if* $I(A:D\mid B)_\omega \le \varepsilon$ *and* $\widetilde{\omega}_\varepsilon \in \mathfrak{M}_{A:B:D}$ *is chosen so that*

$$\|\omega - \widetilde{\omega}_\varepsilon\|_1 \le \delta^{\mathrm{M}}_{A:B:D}(\varepsilon),$$

the corresponding exact splice $\widetilde{\omega}'_\varepsilon$ *satisfies*

$$\left|\operatorname{Tr}(X\omega) - \operatorname{Tr}(X\widetilde{\omega}'_\varepsilon)\right| \le \|X\|_\infty\, \delta^{\mathrm{M}}_{A:B:D}(\varepsilon)$$

for every observable X *supported on* $A\cup b_L$*.*

Independently, if $I(A:D\mid B)_\omega \le \varepsilon$*, then there exists a recovery map* $\mathcal{R}_{B\to BD}$ *such that*

$$\|\omega_{ABD} - (\mathrm{id}_A \otimes \mathcal{R}_{B\to BD})(\omega_{AB})\|_1 \le 2\sqrt{1-e^{-\varepsilon}} \le 2\sqrt{\varepsilon}.$$

Proof. The exact splice statement is the usual blockwise factorization argument:

$$\operatorname{Tr}(X\rho'_{ABD'}) = \sum_\alpha p_\alpha \operatorname{Tr}\Big(X\,\rho^{(\alpha)}_{Ab^\alpha_L}\Big) \operatorname{Tr}\Big(\sigma^{(\alpha)}_{b^\alpha_R D'}\Big).$$

Each right factor is normalized, so the value agrees with the same computation for ρ_{ABD}.

For the controlled statement, compactness of the fixed finite-dimensional state space and continuity of conditional mutual information imply $\delta^{\mathrm{M}}_{A:B:D}(\varepsilon) \to 0$: otherwise one could find a sequence with $I(A:D\mid B) \to 0$ staying a fixed trace distance away from every exact Markov state, contradicting convergence of a subsequence to an exact Markov limit point. Once $\widetilde{\omega}_\varepsilon$ is chosen, the exact splice identity for $\widetilde{\omega}_\varepsilon$ gives

$$\left|\operatorname{Tr}(X\omega) - \operatorname{Tr}(X\widetilde{\omega}'_\varepsilon)\right| = \left|\operatorname{Tr}[X(\omega - \widetilde{\omega}_\varepsilon)]\right| \le \|X\|_\infty\, \delta^{\mathrm{M}}_{A:B:D}(\varepsilon).$$

The final inequality is the standard Fawzi–Renner recovery bound [15]. ☐

This appendix therefore uses exact splice identities in only two regimes: literal exact Markovity, or a controlled collar family on one fixed finite-dimensional model for which $\delta^{\mathrm{M}}_{A:B:D}(\varepsilon) \to 0$. Fawzi–Renner recovery supplies the constructive recovered comparison state; the fixed-collar modulus supplies the exact-Markov comparison. Small one-shot conditional mutual information is not silently upgraded to an exact normal form.

B Fixed-Cutoff Realization, Quotient Repair, and Edge Centers

This appendix carries the fixed-cutoff realization and edge-center items for the consensus paper. They sharpen the quotient-first repair interpretation used throughout the consensus paper and make the collar boundary data explicit at the same finite patch-net level. On the declared fixed-cutoff collar branch, the local repair step is read from exact Markov splice or a declared Petz/Fawzi–Renner recovery move on that same collar data; the recovery move gives a recovered comparison state, while exact splice requires exact Markovity or a controlled fixed-collar replacement modulus. Representative repair maps are only lifts of the resulting quotient-local update.

B.1 Quotient Repair and UV Underdetermination

At fixed cutoff, each regulator cell x carries a finite-dimensional factor $\mathfrak{h}_x$, patch algebras are finite type-I algebras, and gauge-as-gluing is realized as a compact boundary redundancy action on cut data. The physical repair law therefore belongs on the overlap-invariant quotient rather than on hidden representatives. If $q : \Sigma \to \Sigma/\Gamma$ is the quotient by boundary redundancy and $\overline{T}_i$ is the physical quotient update, a representative-level map T_i is only required to be a lift satisfying

$$q \circ T_i = \overline{T}_i \circ q.$$

Hence

$$q(T_i(\gamma \cdot s)) = q(T_i(s))$$

for gauge-equivalent inputs. Quotient descent is therefore structural, while strict representative-level covariance is only implementation bookkeeping. The burden is not to postulate a repair rule, but to prove that the accepted recovery-derived local moves satisfy the stated repair-completeness, support-local disjoint-commutation, nested-collar restriction-compatibility, and Petz-domain control clauses on the declared branch. The touched-overlap acceptance contract yields finite Lyapunov descent and derived termination for accepted moves, while the fixed-cutoff gluing package carries the parenthesization-invariant union-collar state used for the local diamond.

Proposition B.1 (Ancilla-stable UV underdetermination)**.** *Let a fixed-cutoff OPH realization be stabilized by finite ancillary factors K_P in a fixed product state, with observable patch algebras embedded as $\mathcal{A}(P) \otimes \mathbf{1}_{K_P}$ and repair dynamics acting trivially on the ancillas. Then observable expectations on the physical subalgebras, overlap data, the local-Gibbs branch, the collar conditional mutual information $I(A : D \mid B)$, the Fawzi–Renner remainder, the collar Markov modulus, and the quotient normal form are unchanged. Thus the fixed-cutoff theorem package determines the UV branch only modulo such ancillary stabilization together with gauge or implementation hiding, not a unique microscopic presentation.*

Proof. Product ancillas leave physical observables unchanged, cancel additively inside conditional mutual information, and are inert under the repair maps. Hence every invariant listed above is unchanged. □

B.2 Derived Boundary Data and Ordinary EC

Proposition B.2 (Derived boundary gluing datum)**.** *Choose a finite regulator chart for the patches meeting along a connected cut Σ. Because the local overlap algebras are finite-dimensional matrix algebras, any overlap-consistent recharting is an inner automorphism and is implemented by a unitary on the cut Hilbert space. The compact closure of the subgroup generated by these recharting* *unitaries is a compact boundary redundancy group K_Σ. If triple-overlap defects are central, the projective composition law lifts to a compact central extension $\widehat{K}_\Sigma$; on the ordinary branch one simply sets $\widehat{K}_\Sigma = K_\Sigma$. A genuinely noncentral 2-group defect is the only obstruction to reducing the overlap transition system to an ordinary compact group action.*

Theorem B.3 (Derived EC decomposition)**.** *Under the fixed-cutoff regulator realization above, and on the ordinary or central-defect branch, the collar Hilbert space is*

$$\mathcal{H}_{B_\delta} = (\tilde{\mathcal{H}}_{B_L} \otimes \tilde{\mathcal{H}}_{B_R})^{\widehat{K}_\Sigma} \cong \bigoplus_\alpha \left(\mathcal{H}_{b_L^\alpha} \otimes \mathcal{H}_{b_R^\alpha}\right),$$

and the center of the collar algebra is generated by the block projectors:

$$Z(\mathcal{A}(B_\delta)) = \bigoplus_\alpha \mathbb{C} \cdot \mathbf{1}_\alpha.$$

The right half-collar carries the contragredient representation because it sees inverse transport across the same cut.

Remark B.4. This is the finite-patch-net origin of the collar center used by the later Markov, record, and observer packages. Exact Markovity is an additional state hypothesis; EC provides the kinematic block structure.

B.3 Higher-Gauge Replacement on the Genuinely Noncentral Branch

Proposition B.5 (Derived higher-gauge cut datum)**.** *On the genuinely noncentral branch, weak overlap gluing on a connected cut Σ is encoded by a compact crossed module*

$$\mathbb{K}_\Sigma = (H_\Sigma \xrightarrow{\partial_\Sigma} G_\Sigma, \triangleright)$$

with defect class

$$q_\Sigma \in \check{H}^2(N_\Sigma, H_\Sigma \to G_\Sigma),$$

and compact higher-gauge change system

$$\mathcal{T}_\Sigma = C^1(N_\Sigma, H_\Sigma) \rtimes C^0(N_\Sigma, G_\Sigma).$$

Theorem B.6 (Higher-gauge EC decomposition and defect transport)**.** *On the genuinely noncentral branch,*

$$\mathcal{H}^{2g}_{B_\delta} = (\tilde{\mathcal{H}}_{B_L} \otimes \tilde{\mathcal{H}}_{B_R})^{\mathcal{T}_\Sigma} \cong \bigoplus_\lambda (\mathcal{H}_{b_L^\lambda} \otimes \mathcal{H}_{b_R^\lambda}),$$

and

$$Z(\mathcal{A}_{2g}(B_\delta)) = \bigoplus_\lambda \mathbb{C} \cdot \mathbf{1}_\lambda.$$

Moreover the higher-gauge defect class q_Σ is invariant under local rechartings, vanishes iff the defect is removable, and classifies fixed-cutoff genuinely noncentral sectors.

Corollary B.7 (Exact Markov adds the state factorization)**.** *On either the ordinary/central branch of the previous theorem or the genuinely noncentral higher-gauge branch, if in addition*

$$I_\omega(A_\delta : D_\delta \mid B_\delta) = 0,$$

or one passes to the explicitly stated idealized recoverability limit that reduces to exact Markovity, then

$$\rho_{A_\delta B_\delta D_\delta} = \bigoplus_\alpha p_\alpha \left(\rho_{A_\delta b_L^\alpha} \otimes \rho_{b_R^\alpha D_\delta} \right).$$

EC therefore gives the kinematic block decomposition, while exact Markovity is the extra state input that gives the HJPW normal form.

Proposition B.8 (Parenthesization-invariant union-collar gluing)**.** *Fix a finite union collar U built from two overlapping local repair collars on the declared fixed-cutoff branch. On the ordinary or central-defect branch, the physical glued state on U determined by the exterior marginals and sector data is independent of how the local gluings are parenthesized, up to the boundary-redundancy action. On the genuinely noncentral branch, the same statement holds up to the crossed-module change system $\mathcal{T}_\Sigma$. Hence the physical quotient-local glued state on U is well defined independently of parenthesization.*

Proof. On the ordinary or central-defect branch, the preceding edge-center decomposition writes the collar algebra as a direct sum of sector blocks generated by central projectors. Reparenthesizing two overlapping local gluings changes only the representative by the central boundary action and cannot move the state between those block projectors, so the quotient-local glued state is unchanged. On the genuinely noncentral branch, the failure of strict composition is recorded by the crossed-module associator, and the higher-gauge theorem above identifies rechartings exactly with the $\mathcal{T}_\Sigma$-coboundary action. Reparenthesization therefore changes only the representative inside one $\mathcal{T}_\Sigma$-orbit. In either case the physical quotient state is parenthesization-independent. ☐

Remark B.9. Approximate recoverability gives controlled deviations from this normal form; it is not implied by EC alone. This is the fixed-cutoff topological package behind the consensus paper's quotient and record-language surface.

Corollary B.10 (Physical observables are invariant on one quotient-local glued state)**.** *Let U be a finite union collar on the declared fixed-cutoff branch, and let ω_U, ω'_U be two microscopic representatives of the same quotient-local glued state from Proposition B.8. Then every physical observable X on the collar fixed-point / quotient-local algebra has the same expectation in both representatives:*

$$\mathrm{Tr}(X\omega_U) = \mathrm{Tr}(X\omega'_U).$$

In particular the same holds for the central sector projectors and for any observer-accessible record observable generated from them on that same declared surface.

Proof. On the ordinary or central-defect branch, Proposition B.8 says the two representatives differ only by the boundary-redundancy action inside one fixed sector block. The fixed-point collar algebra and its central block projectors are invariant under that action, so their expectation values agree. On the genuinely noncentral branch, the same proposition says the two representatives differ only inside one $\mathcal{T}_\Sigma$-orbit, and the quotient-local physical algebra $\mathcal{A}_{\mathrm{phys}}(U)$ of Definition 5.5 is defined precisely on that orbit space. Therefore the induced physical state and all expectations of physical observables agree there as well. ☐

C Assumption-Dependent Distributed-Systems and QECC Extensions of the Consensus Formalism

Support labels.

[**Established**] Follows from cited prior work or a complete argument given here.

[**Assumption-dependent**] True under additional assumptions not derived from OPH first principles.

[**Conjecture**] A plausible open direction, not a settled result.

B.1 Theorem 1: QBFT Safety Bound

Definition C.1 (QBFT-style protocol)**.** *A consensus protocol is* QBFT-style *in this analysis if it satisfies the following three structural properties. The safety proof of Theorem C.2 uses all three; the theorem does not hold for protocols lacking any of them without a compensating change to the argument.*

(P1) ***One-vote-per-view.*** *Each nonfaulty node casts at most one vote per view number. A node that has voted in view v ignores any later request to vote in view v.*

(P2) ***Certificate semantics.*** *A decision requires a valid quorum certificate: $2f + 1$ distinct, unforgeable, authenticated votes for the same value in the same view.*

(P3) ***DLS-style view-change.*** *If no certificate is produced within a timeout, every nonfaulty node increments the view number by one and a new leader is selected by a fixed deterministic rule. At GST, timeouts fire correctly and the view-change terminates in bounded rounds.*

The Istanbul BFT / QBFT protocol family [6, 7] satisfies (P1)–(P3) and is the intended instance.

Assumptions A1–A6.

(A1) **Partial synchrony (DLS).** Fixed but initially unknown bounds Δ (message delay) and Φ (processing rates). *Safety* holds without extra timing assumptions; *liveness* holds after the Global Stabilisation Time (GST).

(B2) **Byzantine fault model.** At most f observers behave arbitrarily; the remaining $n - f$ are nonfaulty.

(C3) **Optimal fault bound.** $n \geq 3f + 1$ (necessary: [4]; sufficient: [4]).

(D4) **Strong quorum connectivity.** Every quorum Q with $|Q| = 2f + 1$ is strongly connected within G: for any $u, v \in Q$ there is a directed path in G contained entirely in Q. This is strictly stronger than requiring the overlap graph of quorums to be connected, and is needed to propagate signed votes within a quorum.

(E5) **Message authentication.** All messages carry unforgeable digital signatures.

(F6) **OPH quorum overlap.** Any two quorums Q_a, Q_b of size $2f + 1$ satisfy $|Q_a \cap Q_b| \geq f + 1$ (guaranteed by (A3)).

Theorem C.2 (QBFT Safety Bound [Established under A1–A6])**.** *Under assumptions (A1)–(A6), any consensus protocol satisfying (P1)–(P3) of Definition C.1 and run over the OPH observer graph satisfies:*

(i) ***Safety.*** *No two nonfaulty observers finalise conflicting patch states.*

(ii) ***Liveness.*** *After GST, every nonfaulty observer finalises within $O(f \cdot \Delta)$ wall-clock time.*

(iii) ***Optimality.*** *The bound $f < n/3$ is tight.*

Proof sketch. Safety. Suppose O_a and O_b finalise $s_a \neq s_b$ in the same view. By (P2), each required a certificate of $q = 2f+1$ votes: sets Q_a, Q_b. By (A3): $|Q_a \cap Q_b| \geq (2f+1)+(2f+1)-(3f+1) = f+1$. By (A2), at most f are Byzantine, so $Q_a \cap Q_b$ contains a nonfaulty O^*. By (A4), O^*'s signed vote is path-reachable within both quorums. By (P1), O^* voted for at most one value. Contradiction.

Liveness and Optimality follow from [5] (Thm. 4.4) and [4], cited directly.

Note on FLP. Fischer, Lynch, Paterson [3] is an impossibility result for fully asynchronous systems; it does not bear on achievability under partial synchrony (A1). □

B.2 Theorem 2: Convergence of the OPH Repair Map

Definition C.3 (OPH Repair Map: Petz form)**.** *Let $\sigma \in \mathcal{D}(\mathcal{H})$ be a full-rank reference state and $\mathcal{N} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ a quantum channel. The* OPH repair map *is*

$$\mathcal{R}_{\sigma,\mathcal{N}}(\rho) := \sigma^{1/2}\, \mathcal{N}^{\dagger}(\mathcal{N}(\sigma)^{-1/2}\, \rho\, \mathcal{N}(\sigma)^{-1/2})\, \sigma^{1/2},$$

where $\mathcal{N}^{\dagger}$ is the adjoint channel and inverses are taken on $\mathrm{supp}(\mathcal{N}(\sigma))$*.*

Remark C.4 (Petz map vs. trace-distance projection)*.* The closest-point trace-distance projection $\mathcal{P}_{\mathcal{S}}(\rho) := \arg\min_{\tau \in \mathcal{S}} \frac{1}{2}\|\rho - \tau\|_1$ is a different object from the Petz map: it is defined by a variational problem in trace-norm geometry and is not CPTP in general. The two coincide only in very special cases not automatic in the OPH setting. All subsequent properties refer exclusively to Definition C.3.

Proposition C.5 (Petz map CPTP: domain-restricted statement [Established, subject to domain restriction])**.** *Let σ have full support on $\mathcal{H}$.*

(a) *$\mathcal{R}_{\sigma,\mathcal{N}}$ is completely positive.*

(b) *$\mathcal{R}_{\sigma,\mathcal{N}}$ is trace-preserving on* $\mathrm{supp}(\mathcal{N}(\sigma))$*, i.e., on inputs ρ for which $\mathcal{N}(\sigma)^{-1/2}\rho\mathcal{N}(\sigma)^{-1/2}$ is well-defined.*

(c) *If additionally $\mathcal{N}(\sigma)$ has full rank on $\mathcal{K}$, then $\mathcal{R}_{\sigma,\mathcal{N}}$ is CPTP on all of $\mathcal{B}(\mathcal{K})$.*

If $\mathcal{N}(\sigma)$ is not *full rank on $\mathcal{K}$, then either (i) the domain must be restricted to* $\mathrm{supp}(\mathcal{N}(\sigma))$*, or (ii) pseudoinverses must replace the inverses (generalised Petz map; cf. [10]), or (iii) a regularisation $\mathcal{N}(\sigma) \mapsto \mathcal{N}(\sigma) + \varepsilon\mathbf{1}$ must be introduced. Note that full-rank σ does* not *prevent $\mathcal{N}(\sigma)$ from being rank-deficient: the channel may map the support of σ into a strict subspace of $\mathcal{K}$. In the OPH setting, whether $\mathcal{N}(\sigma)$ is full rank depends on the specific overlap channel and must be verified for the chosen analytic channel model. The finite OPH repair theorem instead uses the declared Lyapunov descent law on a finite patch net.*

Proof. Complete positivity follows from composing three CP operations:

(i) sandwiching by $\mathcal{N}(\sigma)^{-1/2}(\cdot)\mathcal{N}(\sigma)^{-1/2}$ on $\mathrm{supp}(\mathcal{N}(\sigma))$;

(ii) $\mathcal{N}^{\dagger}$;

(iii) sandwiching by $\sigma^{1/2}(\cdot)\sigma^{1/2}$.

Trace preservation in the full-rank case: Petz [8]; Fagnola–Umanità [9]. □

Proposition C.6 (Analytic contraction certificate [Assumption-dependent])**.** *Suppose a declared quotient repair map $T : Q \to Q$ on a metric physical quotient (Q, d_Q) is strictly contractive with coefficient $\lambda \in (0, 1)$:*

$$d_Q(Tx, Ty) \leq \lambda d_Q(x, y).$$

Then T has at most one fixed point. If a fixed point $x_\star$ exists, the ideal iterates obey $d_Q(T^t x, x_\star) \leq \lambda^t d_Q(x, x_\star)$. This is an analytic contraction condition. It is not required for the finite OPH normal-form theorem, where termination follows from strict Lyapunov descent of accepted repairs.

Theorem C.7 (Noisy approximate repair stability [Contraction branch])**.** *Assume the contraction certificate of Proposition C.6, and let $\widetilde{T} : Q \to Q$ be an implemented noisy repair map satisfying*

$$d_Q(\widetilde{T}x, Tx) \leq \varepsilon \qquad \text{for all } x \in Q.$$

If $x_\star$ is the ideal fixed point of T, then

$$d_Q(\widetilde{T}^t x, x_\star) \leq \lambda^t d_Q(x, x_\star) + \frac{\varepsilon}{1 - \lambda}.$$

Thus the noisy branch converges only to a controlled error ball, and only after the contraction certificate and uniform implementation-error bound are supplied.

Proof. The recursion

$$d_Q(\widetilde{T}x, x_\star) \leq d_Q(\widetilde{T}x, Tx) + d_Q(Tx, Tx_\star) \leq \varepsilon + \lambda d_Q(x, x_\star)$$

iterates to the displayed geometric-series bound. □

Proposition C.8 (Spectral-gap criterion [Model-dependent])**.** *Let $\mathcal{T}$ be the Markov, channel, or transfer operator induced by iterated OPH repair on a declared analytic realization. If $\mathcal{T}$ has stationary projection $\Pi_\star$ and constants $C < \infty$, $\delta > 0$ such that on the nonstationary subspace*

$$\|\mathcal{T}^t - \Pi_\star\| \leq Ce^{-\delta t},$$

then the corresponding analytic channel model has exponential convergence. The finite OPH repair package supplies termination by Lyapunov descent on its declared finite state space; a spectral gap is a separate quantitative mixing condition for this BFT/QECC-style extension.

Theorem C.9 (Exponential Convergence [Under Proposition C.8])**.** *Under Proposition C.8, for any initial analytic state ρ,*

$$\|\mathcal{T}^t \rho - \Pi_\star \rho\| \leq C\, e^{-\delta t} \|\rho - \Pi_\star \rho\|.$$

This theorem belongs only to the spectral-gap branch. It is not a consequence of a bare finite overlap graph or of finite Lyapunov descent alone.

B.3 Theorem 3: QECC Correspondence

Notation. $N = \dim(\mathcal{H}) = 2^n$ for n physical qubits. Standard notation: $[[n, k, d]]$ stabilizer code; $K = 2^k$; quantum Singleton bound: $k \leq n - 2(d - 1)$.

Theorem C.10 (No free min-cut theorem for bare overlap graphs [Established])**.** *Let $G = (V, E)$ be any connected graph with $|V| \geq 2$. The graph G alone does not determine the Hamming distance of the consistency set C of a finite overlap net on G. In particular, the same graph can realize a binary constraint code of distance 1 or a binary repetition code of distance $|V|$.*

Proof. Set $S_i = \{0,1\}$ for every vertex. For the repetition realization, choose $I_e = \{0,1\}$ and let both endpoint readouts be the identity. Then every edge imposes $x_i = x_j$, so the only global codewords are $00\cdots0$ and $11\cdots1$, whose Hamming distance is $|V|$.

For the trivial-overlap realization, keep the same vertex state spaces but let every endpoint readout be the constant map to 0. Then every binary assignment is globally consistent, so the minimum Hamming distance among distinct codewords is 1. The graph is unchanged. Therefore distance is a property of the code realization–state spaces, readout maps, logical dictionary, metric, and error model–not of the bare overlap graph. □

Definition C.11 (Topological-code realization certificate)**.** *An OPH overlap network may be treated as a QECC/topological code only after supplying a tuple*

$$\mathsf{TCert} = (K, \mathcal{H}_{\mathrm{phys}}, \mathcal{H}_{\mathrm{code}}, \partial_2, \partial_1, S_X, S_Z, \mathcal{L}_X, \mathcal{L}_Z, \mathcal{E}, \mathcal{R}),$$

where $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ is a chain complex over $\mathbb{F}_2$, the physical carriers live in $\mathcal{H}_{\mathrm{phys}}$, the protected subspace is $\mathcal{H}_{\mathrm{code}}$, S_X, S_Z are stabilizer or gauge checks, $\mathcal{L}_X, \mathcal{L}_Z$ are logical-operator classes, $\mathcal{E}$ is a declared error family, and $\mathcal{R}$ is a recovery map or recovery family.

Theorem C.12 (Certified topological-code distance and min-cut [Assumption-dependent])**.** *Suppose a certificate* TCert *of Definition C.11 is supplied, with logical classes identified as*

$$\mathcal{L}_X \simeq H_1(K;\mathbb{F}_2), \qquad \mathcal{L}_Z \simeq H^1(K;\mathbb{F}_2),$$

and with boundary conditions excluding lower-weight trivial representatives. Then the certified distance is

$$d = \min\left\{ \min_{\ell \in \mathcal{L}_X \setminus 0} |\ell|, \min_{\ell^\star \in \mathcal{L}_Z \setminus 0} |\ell^\star| \right\}.$$

Only in geometries where this homological systole equals the relevant graph min-cut may one write $d = \mathrm{mincut}(G_{\mathrm{OPH}})$ [12, 13].

Proof. This is the standard stabilizer/topological-code distance statement once the chain complex, checks, logical representatives, and boundary conditions are declared. The min-cut equality is an additional geometric identification of that homological minimum with a graph cut. By Theorem C.10, it cannot be inferred from the bare graph. □

Conjecture C.13 (Communication complexity [Conjecture])**.** *The OPH consensus-repair protocol, realised as a quantum communication task, has per-round complexity $O(n \cdot \mathrm{poly}(d))$ for a chosen communication encoding (cf. [14]). The fixed finite repair theorem gives termination after a supplied descent law; it does not by itself fix a quantum communication complexity class for every implementation.*

Theorem C.14 (QECC resilience under a supplied code certificate [Assumption-dependent])**.** *Assume a genuine code subspace $\mathcal{H}_{\mathrm{code}} \subseteq \mathcal{H}_{\mathrm{phys}}$ with projector Π, and let $\mathcal{E}_t$ be the declared set of errors supported on fewer than t corrupted patches or physical carriers. If*

$$\Pi E_a^\dagger E_b \Pi = \alpha_{ab} \Pi \qquad \forall E_a, E_b \in \mathcal{E}_t,$$

then there exists a recovery channel correcting all errors in $\mathcal{E}_t$ [11]. If the supplied certificate also gives distance d, then all errors of weight $t < d/2$ are correctable. No such resilience statement follows from the bare overlap graph.

Corollary C.15 (QECC extension inventory)**.** *Under Definition C.11, Theorem C.12, and Theorem C.14, the OPH BFT/QECC extension carries the following status split:*

(i) [Assumption-dependent] Code distance is the certified homological minimum; it equals $\mathrm{mincut}(G_{\mathrm{OPH}})$ *only under the additional systole/min-cut identification.*

(ii) [Established] The Knill–Laflamme QECC theorem supplies recovery once the projector and error family satisfy the displayed condition.

(iii) [Conjecture] Per-round communication complexity is $O(n \cdot \mathrm{poly}(d))$*.*

B.4 Theorem 4: Asynchronous Convergence

Why fairness alone does not give a probability-1, spectral, or wall-clock statement. Standard strong fairness guarantees that every enabled action fires infinitely often along any fair schedule; it does not impose a probability space on schedules, a transfer-operator spectral gap, or a message-delay bound. A convergence statement of the form "converges with probability 1" requires a measure on schedules. Exponential convergence requires the spectral-gap certificate of Theorem C.9. Bounded wall-clock liveness requires partial synchrony and quorum assumptions. The FLP impossibility result [3] confirms that fairness is insufficient for bounded-time consensus in a fully asynchronous system.

Additional assumptions for a quantitative bound.

(B1) Finite known bound Δ on message delay after GST.

(B2) Finite bound Φ on processing rates.

(B3) $f < n/3$.

Theorem C.16 (Eventual finite repair termination [Finite-descent branch])**.** *For a finite OPH patch net with a strict Lyapunov-decreasing accepted repair relation, every maximal repair run terminates after finitely many accepted repairs. The step count is bounded by the finite value-set bound of Proposition 3.12. If the local-diamond and repair-completeness clauses also hold, Newman's lemma upgrades this termination statement to the unique schedule-independent quotient normal form of Theorem 3.15.*

This theorem is finite descent convergence in repair steps. It is not trace-norm convergence of a Petz channel, not probability-one convergence over random schedules, not exponential convergence, and not a wall-clock liveness theorem.

Theorem C.17 (Quantitative Convergence [Assuming (B1)–(B3)])**.** *In a partially synchronous OPH observer network satisfying (B1)–(B3), after GST every nonfaulty observer reaches consensus within* $T = O(f \cdot \Delta)$ *wall-clock time (by applying the DLS framework [5], Thm. 4.4, to the OPH repair protocol; requires (B1) and (B2) explicitly and does not follow from fairness alone).*

B.5 Extension Boundaries

- A bare OPH overlap graph is a finite constraint-code presentation only. Its graph does not determine code distance, correctable error weight, or min-cut resilience.

- Analytic spectral-gap and full-rank estimates for a chosen stochastic or channel-level BFT/QECC realization are model-specific refinements. They are separate from the finite OPH repair theorem, where the accepted repair law supplies Lyapunov descent directly.
- Long-run noisy approximate consensus is available only on the fair-block contraction branch of Theorem 8.1: the chosen implementation must certify fair blocks, expected contraction toward the exact quotient normal-form set, and controlled within-block excursions. Fairness alone does not supply that certificate.
- Topological-code distance equals graph min-cut only after a concrete chain complex, boundary condition, logical-operator dictionary, error family, and systole/min-cut identification are supplied for the chosen code realization.
- Communication-complexity bounds require a concrete quantum communication encoding and implementation cost model. They are not consequences of finite normal-form termination alone.
- The core OPH consensus paper supplies observable-level confluence and refinement-limit normal-form/holonomy classes on their declared theorem surfaces; the BFT/QECC statements above are separate protocol-style extensions.

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