The Positive-Chamber Koide Identity for Icosahedral Face Circulants

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Abstract

The stabilizer of an oriented icosahedral face acts regularly on its three corners. Every Hermitian response commuting with this cyclic action is a circulant $C = aI + bR + \bar{b}R^2$, $R^3 = I$. In the chamber where all three eigenvalues are nonnegative and may be read as square-root masses, its Koide invariant is

$$Q = \frac{1}{3} + \frac{2}{3}\left(\frac{|b|}{a}\right)^2.$$

Thus $Q = 2/3$ is exactly equivalent to the single balance condition $|b|/a = 1/\sqrt{2}$. The phase of b drops out of Q and continues to control the two mass ratios jointly. A finite tracial Gelfand–Naimark–Segal construction with equal rank-two event blocks gives the balanced modulus as a conditional theorem. The local face fiber has no constructed physical chiral-family attachment, so the result is a representation-theoretic identity and a finite conditional implication, without a charged-lepton mass prediction. The numerical response model associated with this construction was historically target-informed; its near-equality with the Particle Data Group central Koide coordinate is therefore a retrospective diagnostic.

Keywords: Koide relation, circulant matrices, cyclic symmetry, icosahedral symmetry, finite event algebra, GNS representation

1 The face-corner carrier

Koide's charged-lepton relation [1] is usually written in terms of the three positive square-root masses. The construction below identifies the exact condition it imposes on a cyclic Hermitian response.

Let $G = A_5$ act by proper rotations on the icosahedron. The twenty outward oriented faces form the transitive orbit G/C_3. The stabilizer of one face cyclically permutes its three corners, so every face carries a local copy of the regular C_3 representation. If R denotes the cyclic shift, then

$$R^3 = I, \qquad R^\dagger = R^2.$$

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The commutant of this regular action is the three-dimensional circulant algebra

$$\{x_0 I + x_1 R + x_2 R^2 : x_j \in \mathbb{C}\}.$$

Hermiticity restricts a response to

$$C = aI + bR + \bar{b}R^2, \qquad a \in \mathbb{R}, \quad b \in \mathbb{C}. \tag{1}$$

Write $b = \rho e^{i\delta}$, with $\rho = |b| \geq 0$. Fourier diagonalization gives

$$\lambda_k = a + 2\rho \cos\left(\delta + \frac{2\pi k}{3}\right), \qquad k = 0, 1, 2. \tag{2}$$

The unordered spectrum is independent of the chosen face representative. This is a local bundle statement. The sixty face-corner flags form the regular A_5 torsor; the geometry alone supplies no canonical global three-dimensional physical family space.

2 The positive-chamber identity

Theorem 1 (Positive-chamber Koide identity)**.** *Let C be the Hermitian circulant in Eq. (1), with $a > 0$. Suppose its three eigenvalues in Eq. (2) are nonnegative and, for one $s > 0$,*

$$\sqrt{m_k} = \sqrt{s}\lambda_k.$$

Then

$$Q := \frac{\sum_{k=0}^{2} m_k}{\left(\sum_{k=0}^{2} \sqrt{m_k}\right)^2} = \frac{1}{3} + \frac{2}{3}\left(\frac{\rho}{a}\right)^2. \tag{3}$$

Consequently,

$$Q = \frac{2}{3} \iff \frac{\rho}{a} = \frac{1}{\sqrt{2}}. \tag{4}$$

Proof. Set $c_k = \cos(\delta + 2\pi k/3)$. The roots-of-unity identities give

$$\sum_{k=0}^{2} c_k = 0, \qquad \sum_{k=0}^{2} c_k^2 = \frac{3}{2}.$$

It follows that

$$\sum_k \lambda_k = 3a, \qquad \sum_k \lambda_k^2 = 3a^2 + 6\rho^2.$$

The scale s cancels from Q, yielding Eq. (3). Since $\rho/a \geq 0$, Eq. (3) equals $2/3$ exactly at $\rho/a = 1/\sqrt{2}$. □

Remark 2 (The chamber is part of the theorem). The signed trace calculation holds algebraically for every δ. Physical square roots use $|\lambda_k|$ when an eigenvalue is negative, and then the denominator is not the signed trace. At $\rho/a = 1/\sqrt{2}$, the positive chamber is

$$|\delta| \leq \frac{\pi}{12} \pmod{\frac{2\pi}{3}}.$$

Theorem 1 makes no physical Koide statement outside that chamber.

Remark 3 (What the phase contains). Inside the positive chamber, Q is independent of δ. The three eigenvalues, and hence the two reported mass ratios, vary with δ. The balance condition removes one scale-free degree of freedom from a three-mass spectrum. It does not determine the ratios.

3 Conditional finite tracial balance

The balance condition in Eq. (4) can arise exactly from a finite event packet. This subsection states the packet as an independent conditional theorem.

Let

$$\mathcal{V} = \mathbf{1} \oplus \chi \oplus \overline{\chi}, \qquad \mathcal{H}_{\mathrm{or}} = \mathbb{C}^2, \qquad \mathcal{A} = B(\mathcal{V} \otimes \mathcal{H}_{\mathrm{or}}) \simeq M_6(\mathbb{C}).$$

Here $\mathbf{1}$ is the neutral cyclic mode, $\chi \oplus \overline{\chi}$ is the two-dimensional charged plane, and $\mathcal{H}_{\mathrm{or}}$ is a two-state orientation record. Let P_0 and P_c project onto the neutral line and charged plane. For a rank-one oriented event q_+, define

$$Z_0 = P_0 \otimes I_{\mathrm{or}}, \qquad Z_c = P_c \otimes q_+, \qquad E_+ = Z_0 + Z_c.$$

Both Z_0 and Z_c have rank two.

Theorem 4 (Conditional finite tracial-GNS balance)**.** *Condition the normalized trace state $I_6/6$ on E_+. Map the orthonormal cyclic basis of the resulting square-root amplitude to (I, R, R^2) in $L^2(M_3(\mathbb{C}), \tau_3)$. Then*

$$p_0 = p_c = \frac{1}{2}, \qquad a = \sqrt{p_0}, \qquad \rho = \sqrt{\frac{p_c}{2}},$$

and therefore

$$\frac{\rho}{a} = \frac{1}{\sqrt{2}}, \qquad Q = \frac{2}{3}$$

whenever the positive square-root-mass reading of Theorem 1 is supplied.

Proof. Normalized-trace conditioning assigns probability proportional to block rank. Both accepted blocks have rank two, so $p_0 = p_c = 1/2$. In $L^2(M_3(\mathbb{C}), \tau_3)$, the operators I, R, R^2 are orthonormal. The canonical square-root vector therefore has neutral coefficient $\sqrt{p_0}$ and two charged coefficients of modulus $\sqrt{p_c/2}$. Their ratio is $1/\sqrt{2}$. Theorem 1 then gives $Q = 2/3$ on the positive chamber. $\square$

Theorem 4 is finite and exact. Its event algebra, block choice, conditioned trace, and cyclic response map are premises. A physical charged-lepton statement additionally requires a quotient-visible map from the local face bundle to one chiral three-family response, preservation of the relevant block powers, and a mass readout. Those objects are absent from the theorem.

4 Numerical diagnostic and provenance

Using the Particle Data Group 2026 central masses [2],

$$(m_e, m_\mu, m_\tau) = (0.51099895069,\ 105.6583755,\ 1776.93)\ \mathrm{MeV},$$

gives

$$Q_{\mathrm{PDG}} = 0.6666644634026367.$$

The declared minimal complete public response model gives

$$\begin{aligned} Q_{\mathrm{MCPR}} &= 0.6666644634090389, \\ \rho/a &= 0.7071044442750720, \\ \frac{\rho/a - 1/\sqrt{2}}{1/\sqrt{2}} &= -3.3049 \times 10^{-6}. \end{aligned}$$

The response dimensions, path table, amplitude, phase, and determinant exponent were historically selected with knowledge of the charged-lepton target. The executable evaluation reads no charged target at runtime, yet that runtime separation does not make the model blind or source-derived. No significance or prospective evidence claim follows from the proximity of the two displayed Q values.

Statement	Status
Eq. (3)	exact algebra in the nonnegative-eigenvalue chamber
Theorem 4	exact implication conditional on the declared finite event packet
Icosahedral face fiber	exact local A_5/C_3 bundle and unordered spectrum
Physical family attachment	open
Phase and numerical ratios	open
MCPR numerical proximity	historically target-informed retrospective diagnostic

5 Why residual symmetry does not finish the ratios

The remaining family-shape problem can be stated in the real five-dimensional representation

$$W_5 \simeq \mathrm{Sym}_0^2(\mathbb{R}^3),$$

the traceless symmetric 3×3 matrices with action $A \mapsto gAg^T$.

Proposition 5 (Residual-stabilizer boundary)**.** *Invariance under a threefold or fivefold rotation about an axis n forces*

$$A = \alpha(nn^T - I/3),$$

which has a double eigenvalue. Invariance under a twofold rotation has a three-dimensional linear fixed locus, hence two projective parameters, and that locus admits simple spectrum.

Proof. Choose the rotation axis as the third coordinate. Decompose a traceless symmetric matrix into a planar scalar, planar spin-two anisotropy, transverse vector, and axial entry. Rotation by θ acts on the anisotropy by 2θ and on the transverse vector by θ. For $\theta = 2\pi/3$ or $2\pi/5$, neither block has a nonzero fixed vector, so only the axial combination remains. For $\theta = \pi$, the planar symmetric block survives:

$$A = \begin{pmatrix} u & v & 0 \\ v & w & 0 \\ 0 & 0 & -u-w \end{pmatrix}.$$

This space has dimension three and contains matrices with three distinct eigenvalues. □

Threefold and fivefold symmetry cannot produce three distinct masses. Twofold symmetry leaves two scale-free coordinates, exactly enough to carry the two mass ratios without selecting them. A numerical spectrum therefore requires a specific invariant potential derived from the screen dynamics. This proposition locates the missing input; it supplies no replacement fit.

6 Formal proof boundary

The algebraic core has a Lean 4 formalization. It proves the root sum, square sum, quotient formula, reciprocal-square-root identity, and the nonnegative-modulus equivalence

$$\frac{1}{3} + \frac{2}{3}r^2 = \frac{2}{3} \quad \Longleftrightarrow \quad r = \frac{1}{\sqrt{2}}.$$

The formal module deliberately does not identify signed eigenvalues with physical square roots. Positivity of all three eigenvalues, the phase selection, and the source-to-charged-family attachment remain outside its statement. This division matches Theorem 1: the checked algebra is exact, and the physical reading is a separate premise.

7 Conclusion

The regular three-corner face carrier turns the Koide relation into one transparent modulus condition. In the positive chamber,

$$Q = \frac{2}{3} \quad \Longleftrightarrow \quad \rho/a = \frac{1}{\sqrt{2}}.$$

The finite tracial event packet supplies this balance conditionally through equal block weights and the canonical square-root representation. The result does not determine the phase or mass ratios and does not attach the local face fiber to physical charged leptons. Its durable content is an exact circulant identity, an exact finite conditional balance theorem, and a sharp statement of the remaining physical work.

References

[1] Y. Koide, *New View of Quark and Lepton Mass Hierarchy*, Physical Review D **28**, 252 (1983). https://doi.org/10.1103/PhysRevD.28.252

[2] Particle Data Group, *Review of Particle Physics: 2026 particle listings*, 2026. https://pdg.lbl.gov/2026/listings/particle_properties.html